Details at `arXiv:1307.5812`. Everything is dg. Char=0.

1. Motivation: The classical BV formalism

Defn: *Classical field theory* = the study of those PDE determined by "least action" variational principles = geometry of critical loci (in finite-dimensional manifolds).

C.f.: *QFT* = computing $\int(\text{observable})\exp\left(\frac{i}{\hbar}(\text{action})\right)$, with infinite-dimensional domain of integration.

Defn: The *classical BV formalism* is the observation that the *derived* critical locus of any function has a symplectic form of homological degree +1, i.e. a Poisson structure of deg −1. (Convention: differential has deg −1.)

Historical aside: Batalin–Vilkovisky were physicists, working only with $\mathbb{Z}/2$ ("super") gradings. What's called a "BV algebra" in mathematics is not what B–V discovered. It is (almost) the same with $\mathbb{Z}/2$ gradings, but different with $\mathbb{Z}$ gradings. This is Getzler's fault.

Defn: A *Pois$_d$ structure* on a cdga A is a biderivation making $A[1-d]$ into a dgla. (Poisson = Pois$_1$. Grading conventions will be justified later.)

Defn:

dgla : L_∞ :: Pois$_d$: *semistrict homotopy Pois$_d$*

= system of multiderivations on $\mathcal{O}(X)$ making $\mathcal{O}(X)[1-d]$ into L_∞ alg. "semistrict" = don't weaken Leibniz.

Polemical aside: Actual derived critical loci are always *cotangent bundles*, not just Pois$_0$ spaces. So usual BV formalism requires bracket to be *symplectic*, i.e. nondegenerate. Why not work with those? Because of dualities/symmetries/gauge equivalence.

But *symplectic is wrong*. Locally, Poisson = symplectic with parameters, and we know should study geometry in families. Globally, can have rich dualities/etc., so "families of symplectics" isn't good enough: need Poisson.

Cor: Any s.h.Pois$_0$ space should be considered a (generalized) derived critical locus.

New challenge: Find interesting s.h.Pois$_0$ structures on spaces of "fields." Interpret as classical field theory.

Thm (Alexandrov–Kontsevich–Schwarz–Zaboronsky): M is closed oriented d-dim manifold. X is *symplectic* Pois$_d$. Then $\text{Maps}(M_{\text{dR}}, X)$ = derived space of locally constant maps $M \to X$ is symplectic Pois$_0$.

With one lie. It is symplectic, i.e. has 2-form with trivial kernel. But it's ∞-dim. How to invert to Poisson structure? (And see earlier polemical aside.)

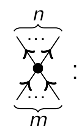
2. Infinitesimal manifolds and dioperads

I have an answer when X = infinitesimal manifold.

Defn: An *infinitesimal manifold (with local coord chart)* is $\operatorname{spec}\widehat{\text{Sym}}(V)$ for a chain complex V. ($\widehat{\text{Sym}}$ = completed symmetric algebra. All geometry should be cont's for power series topology.)

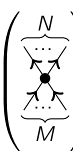
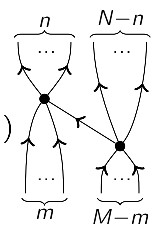
Technical convenience: Let's assume all geometric structures vanish at $0 \in \operatorname{spec}\widehat{\text{Sym}}(V) \approx V^*$. Then a dg structure on $\operatorname{spec}\widehat{\text{Sym}}(V)$ begins with a linear term; absorb it into differential ∂_V making V into chain complex.

Exercise: A s.h.Pois$_d$ structure on $\operatorname{spec}\widehat{\text{Sym}}(V)$ is same as system of tensors



$: V^{\otimes m} \to V^{\otimes n}$ of hom degree $d(m-1)-1$

in $(\text{sign})^{\otimes d} \otimes (\text{triv})$ subrep of $\mathbb{S}_m^{\text{op}} \otimes \mathbb{S}_n \curvearrowright \hom(V^{\otimes m}, V^{\otimes n})$, satisfying

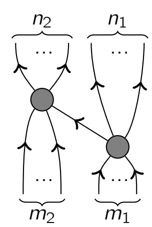
∂_V () $= \sum_{m,n,M-m,N-n \geq 1} (\#)$  $(*)$

Coeffs (#) depend on conventions. Average over permutations of input/output strands, with signs when d = odd.

Defn (Gan):

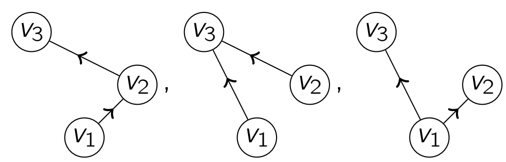
directed trees : *dioperads* :: rooted trees : operads

I.e. a *dioperad* P consists of $\mathbb{S}_m^{\text{op}} \times \mathbb{S}_n$-modules $P(m,n)$ of "m-to-n operations" and *binary compositions*



$: \begin{array}{l} P(m_1, 1+n_1) \otimes P(m_2+1, n_2) \\ \quad \to P(m_1+m_2, n_1+n_2) \end{array}$

satisfying associative axioms for diagrams like:



E.g.: V a chain complex. $\text{End}(V)(m,n) = \hom(V^{\otimes m}, V^{\otimes n})$ defines a dioperad. An *action* of P on V (equivalently, V is a *P-algebra*) is a homomorphism $P \to \text{End}(V)$.

3. Technical tools for working with dioperads

Fact: Dioperads admit model category structure with acylic = quasiisomorphic and fibration = surjection.

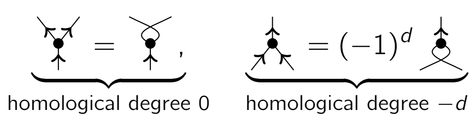
Defn: Give dioperad P satisfying mild conditions, its *bar dual* $\mathbb{D}P$ is freely generated by $P^*[-1]$ with differential dual to $\sum$(binary compositions) : $P^{\otimes 2} \to P$ (extend as derivation; associativity $\Leftrightarrow \partial^2 = 0$.)

E.g.: $(*)$ defines dioperad P_d s.t. P_d-algebras = s.h.Pois$_d$ infinitesimal manifolds. By shape of $(*)$, $P_d = \mathbb{D}(F_d)$ for some F_d. What is it? Read off from $(*)$:

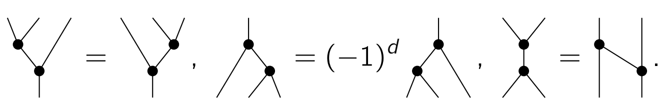
$$F_d(m,n) = (\mathrm{sign})^{\otimes m} \otimes (\mathrm{triv})[d(1-m)], \quad mn > 0.$$

Exercise: An F_d-algebra W is a (noncounital) cocommutative coalgebra plus a (nonunital) commutative algebra on $W[-d]$, plus a compatibility condition. I.e. F_d = Frob$_d$ = *open d-shifted commutative Frobenius*.

Lemma: Frob$_d$ has a presentation with generators:



and *quadratic* relations:



Defn: The *quadratic dual* F^{i} of dioperad F with generators T and relations $R \subseteq T^{\otimes 2}$ is generated by $T^*[-1]$ with relations $R^{\perp}[-2] = \ker((T^*[-1])^{\otimes 2} \to R^*[-2])$.

E.g.: $\mathrm{Frob}_d^{\text{i}} = \mathrm{LB}_d$ controls *d-shifted Lie bialgebras* (i.e. bracket of degree $d-1$, cobracket of degree -1; so V normal Lie bialgebra = $V[-1]$ a LB$_2$-algebra).

Defn: $\exists$ canonical $\mathbb{D}P \twoheadrightarrow P^{\text{i}}$. *Koszul* = this is acyclic.

Fact: Frob$_d$ is Koszul. **Pf:** Uses Lyndon words.

Fact: $\mathbb{D}P$ always cofibrant (=relatively easy to map out of). So P Koszul $\Leftrightarrow \mathbb{D}P \overset{\sim}{\twoheadrightarrow} P^{\text{i}}$ is cofibrant replacelment.

Cor: s.h.Pois$_d$ infinitesimal mans = homotopy LB$_d$ algs.

Fact: $\mathbb{D}\mathbb{D}P \overset{\sim}{\twoheadrightarrow} P$ is always a cofibrant replacement.

Fact: Different cofibrant replacements give homotopy-equivalent notions of "homotopy P-algebra."

Main fact: For any dioperad P (satisfying mild conditions), $\exists$ canonical homomorphism $\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}P \otimes P$.

4. The Poisson AKSZ construction

We want Maps(M_{dR}, X) to be s.h.Pois$_0$ if $X = \operatorname{spec}\widehat{\mathrm{Sym}}(V)$ is s.h.Pois$_d$. First, what is Maps(M_{dR}, X)?

$X \approx V^* \Rightarrow$ Maps$(M_{\mathrm{dR}}, X) \approx \mathcal{O}(M_{\mathrm{dR}}) \otimes V^* \approx \Omega^\bullet_{\mathrm{dR}}(M) \otimes V^* \Rightarrow$ linear fns on Maps$(M_{\mathrm{dR}}, X) \approx (\Omega^\bullet_{\mathrm{dR}}(M) \otimes V^*)^* \approx (\Omega^\bullet(M))^* \otimes V = \mathrm{Chains}_\bullet(M) \otimes V$. Which model of Chains? Depends on how smooth you want "observables" to be.

I.e.: We want $\mathbb{D}\,\mathrm{Frob}_0 \to \mathrm{End}(\mathrm{Chains}_\bullet(M) \otimes V)$ given $\mathbb{D}\,\mathrm{Frob}_d \to \mathrm{End}(V)$ and M oriented.

Cor of Main fact: Since $\mathrm{End}(W \otimes V) = \mathrm{End}(W) \otimes \mathrm{End}(V)$, it suffices to find $\mathbb{D}\mathbb{D}\,\mathrm{Frob}_d \to \mathrm{End}(\mathrm{Chains}_\bullet(M))$, i.e. Chains$_\bullet(M)$ to be homotopy Frob$_d$ algebra. Is it?

Wrong answer: Since M is oriented, $\mathrm{H}_\bullet(M)$ is Frob$_d$. Choose $\mathrm{H}_\bullet(M) \simeq \mathrm{Chains}_\bullet(M)$; use homotopy perturbation theory to transfer Frob$_d$ action on $\mathrm{H}_\bullet(M)$ to homotopy action on Chains$_\bullet(M)$. **Why wrong?** Transferred structure is highly non-local. **Why right?** We do want homotopy Frob$_d$ str to lift Frob$_d$ str on $\mathrm{H}_\bullet$.

Defn: Choose complete metric on M. Operation f : $\mathrm{Chains}_\bullet(M)^{\otimes m} \to \mathrm{Chains}_\bullet(M)^{\otimes n}$ is *quasilocal* if $\exists$ "uv length scale" $\ell \in \mathbb{R}$ s.t. $f(a_1 \otimes \cdots \otimes a_m) \in \mathrm{Chains}_\bullet(B)^{\otimes n}$ where $B = \bigcap_i$(radius-ℓ nbhd of support(a_i)). Triangle inequality $\Rightarrow$ quasilocal ops comprise dioperad QLoc(M).

Remark: QLoc is filtered dioperad. Different metrics give compatible filtrations.

Prop: By defn, QLoc $\curvearrowright$ Chains$_\bullet$. But Chains$_\bullet \hookrightarrow$ Cochains$^{d-\bullet}$ as compactly-supported. And QLoc $\curvearrowright$ Cochains$^{d-\bullet}$ too.

When M not compact, best to use both Chains,Cochains to define Frob$_d$ str that is to be lifted to QLoc.

Thm: $\exists$ canonical contractible space of quasilocal homotopy Frob$_d$-algebra structures on Chains$_\bullet(M)$ lifting Frob$_d$ structure on $(\mathrm{H}_\bullet(M), \mathrm{H}^{d-\bullet}(M))$.

Pf: When ℓ sufficiently small, space of ℓ-quasilocal m-to-n operations has homology $\mathrm{H}^{-\bullet}(M)[d(1-m)]$. Generators of $\mathbb{D}\mathbb{D}\,\mathrm{Frob}_d$ are graded by *syzygy degree* σ. ($\sigma = 0$) generators correspond to operations in Frob$_d$, are in hom deg $d(1-m)$; map them to Thom forms. Obstruction to defining ($\sigma = 1$) generators is difference of Thom forms, hence exact. When $\sigma > 1$, obstruction is in hom degree $> d(1-m)$, hence vanishes.

Defn: The *Poisson AKSZ construction* is the s.h.Pois$_0$ structure on Maps$(M_{\mathrm{dR}}, \operatorname{spec}\widehat{\mathrm{Sym}}(V))$ coming as

$$\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}\mathbb{D}\,\mathrm{Frob}_d \otimes \mathbb{D}\,\mathrm{Frob}_d \to \mathrm{QLoc} \otimes \mathrm{End}(V) \to \mathrm{End}(\mathrm{Chains}_\bullet(M) \otimes V).$$

Remark: Can drop condition that geometry vanishes at $0 \in X$ by using Hirsh–Millès' "curved Koszul duality." Use nonunital but counital Frob. Dual is *traced* Lie bialgebras.

5. Motivation: The quantum BV formalism

Defn: *QFT* = computing $\int(\text{observable})\exp\left(\frac{i}{\hbar}(\text{action})\right)$.

Defn: The *quantum BV formalism* identifies "oscillating integral" with "space X with $\hbar$-dependent second-order diff. op. Δ such that (i) Δ is differential on $\mathcal{O}(X)$, (ii) $\Delta(1) = 0$, (iii) $\Delta|_{\hbar=0}$ is derivation."

So $\Delta|_{\hbar=0}$ is makes $\mathcal{O}(X)$ into cdga. Principal symbol of $\frac{\partial}{\partial\hbar}|_{\hbar=0}\Delta$ is by defn a biderivation. **Exercise:** It makes $\mathcal{O}(X)[-1]$ into dgla. I.e. it is Pois_0 structure on X.

Defn: $(\mathcal{O}(X), \Delta)$ is a *Beilinson–Drinfeld algebra*. Earlier historical aside: "BV algebra" was taken. B–D used this notion in book on CFT. Name due to Costello–Gwilliam.

Defn:

dgla : L_∞ :: BD : *semistrict homotopy BD*

= $\hbar$-dependent differential Δ, vanishing on constants, such that $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$ is $(n+1)$th order diff. op.

Exercise: Princ. symbols of $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$s give s.h.$\mathrm{Pois}_0$ str.

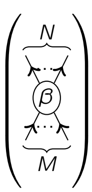
Defn: A *quantization* of a s.h.Pois_0 algebra is a s.h.BD differential with given principal symbols.

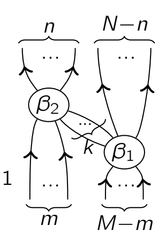
Henceforth, $\hbar$ is formal variable, so $\Delta = \sum \frac{\hbar^n}{n!}\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$.

Exercise: A s.h.BD structure on spec $\widehat{\mathrm{Sym}}(V)$ is same as system of $\mathbb{S}_m^{\mathrm{op}} \otimes \mathbb{S}_n$-invariant degree-$(-1)$ tensors

$: V^{\otimes m} \to V^{\otimes n}$, labeled by $\beta \in \mathbb{N}$

satisfying

∂_V 

$= \sum_{\substack{m,n,\beta_1,\beta_2 \\ \text{s.t. } k=\beta-\beta_1-\beta_2+1\geq 1}}$ 

$(**)$

β is *internal genus*. Eqn $(**)$ is homogeneous for *total genus* = internal genus + genus of diagram.

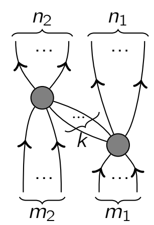
6. Properads

Eqn $(**)$ is not a tree, so dioperads aren't good enough.

Defn (Vallette):

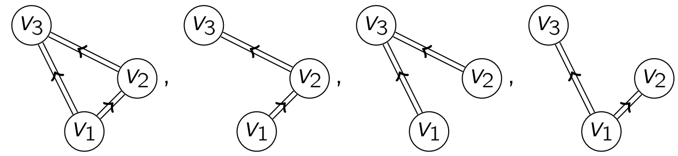
dioperads : directed trees
:: *props* : directed acyclic graphs
:: *properads* : directed *connected* acyclic graphs

I.e. $\mathbb{S}$-bimodule with *binary compositions*



$$: P(m_1, k+n_1) \otimes P(m_2+k, n_2) \to P(m_1+m_2, n_1+n_2)$$

for $k \geq 1$, satisfying associative axioms for diagrams like:



Fact: Properads and props also have model category structures with acyclic=quasiiso and fibration=surjection.

Theorem (Vallette): The functors

Free : {properads} $\leftrightarrows$ {props} : Forget

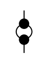
are exact.

Warning: {dioperads} $\leftrightarrows$ {properads} are not exact!

So for dioperad P, "homotopy P-algebra" can depend on whether cofibrant replacement is computed in dioperads or properads/props.

Fact: Properads have bar duals, Koszulity. (Props don't.)

Fact: LB_d is Koszul. Quadratic dual is $\mathrm{invFrob}_d$, controls *involutive* ( $= 0$) d-shifted open Frob algs. s.h.Pois_d remains a cofibrant replacement.

Warning: When d is odd,  $= 0$ automatically. But when d is even, properadic $\mathrm{Frob}_d(m,n) = \mathbb{Q}[x]$, graded by internal genus. When d is even, Frob_d is not known to be Koszul.

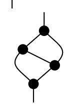
Remark: By eqn $(**)$, s.h.BD infinitesimal manifolds are $\mathbb{D}\,\mathrm{Frob}_0$-algebras. Abstract nonsense still gives $\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}P \otimes P$, now with properadic $\mathbb{D}$.

Question: Is there a *quantum* AKSZ construction? I.e. $\exists\ \mathbb{D}\mathbb{D}\,\mathrm{invFrob}_d \to \mathrm{QLoc}(M)$ for M oriented d-dim?

Formula for $\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}P \otimes P$ is a "sum over diagrams." So this would be a "path integral" quantization.

Answer: One obvious obstruction is involutivity.

Thm: No! When $M = \mathbb{R}$, $\nexists$ properadic homotopy Frob_1 action on $\mathrm{Chains}_\bullet(\mathbb{R})$ sending ,  to Thom forms.

Pf: Use $\mathbb{D}\,\mathrm{LB}_1$. Obstruction dual to  is $-\frac{1}{12}$, which is not exact. Details at `arXiv:1308.3423`.

7. Homological perturbation and Feynman diagrams

Fact (Homological perturbation lemma):

Suppose given a *retraction* (in any additive category)

$$(H_\bullet, \partial_H) \underset{\phi}{\overset{\iota}{\leftrightarrows}} (V_\bullet, \partial) \circlearrowright \eta \qquad \begin{array}{l} \iota\phi = \mathrm{id}_H \\ \phi\iota = \mathrm{id}_V - [\partial, \eta] \end{array}$$

and a *perturbation* $\partial \rightsquigarrow \partial + \delta$ with $(\partial + \delta)^2 = 0$. *If* $(\mathrm{id}_V - \delta\eta)$ *is invertible*, get new retraction:

$$(H_\bullet, \tilde\partial_H) \underset{\tilde\phi = (\mathrm{id}-\eta\delta)^{-1}\phi}{\overset{\tilde\iota = \iota(\mathrm{id}-\delta\eta)^{-1}}{\leftrightarrows}} (V_\bullet, \partial + \delta) \circlearrowright \tilde\eta = \eta(\mathrm{id}-\delta\eta)^{-1}$$

with $\tilde\partial_H = \partial_H + \iota(\mathrm{id} - \delta\eta)^{-1}\delta\phi$. Note: $(\mathrm{id} - \eta\delta)^{-1} = \mathrm{id} + \eta(\mathrm{id} - \delta\eta)^{-1}\delta$. **Pf:** Check some eqns.

Cor: Consider oscillating measure $\mu = \exp(\frac{i}{\hbar}s)\mathrm{d}x_1 \dots \mathrm{d}x_n$, with $s = a\frac{x^2}{2} + b(x)$, matrix a invertible, and b cubic+higher. Stationary phase: if f only supported near 0, then mod $O(\hbar^\infty)$, can work in $V_0 = \mathbb{R}[\![x_1, \dots, x_n]\!] = \widehat{\mathrm{Sym}}(\mathbb{R}^n)$. BV formalism in this case is: $\int f\mu$ depends only on homology class of f in chain complex $V = \mathbb{R}[\![x_1, \dots, x_n, \xi_1, \dots, \xi_n]\!]$, where $\deg(\xi_i) = 1$, with differential $\partial = \partial_a + \partial_b + \hbar\partial_\Delta$, where $\partial_a =$ linear differential on $\widehat{\mathrm{Sym}}(\mathbb{R}^n \overset{a}{\to} \mathbb{R}^n)$, $\partial_b = \vec\nabla_x(b) \cdot \vec\nabla_\xi$, and $\partial_\Delta = \vec\nabla_x \cdot \vec\nabla_\xi$.

Since a is invertible, can choose linear retraction $\mathbb{R} \simeq \widehat{\mathrm{Sym}}(\mathbb{R}^n \overset{a}{\to} \mathbb{R}^n)$ with homotopy $\eta = \frac{1}{p}a^{-1}$ on degree-p polynomials. Then apply HPL: find that $\frac{\int f\mu}{\int \mu} = \mathrm{ev}_0 \circ (\mathrm{id} - (\partial_b + \hbar\partial_\Delta)\eta)^{-1}(f) = \mathrm{ev}_0 \sum_{k=0}^\infty (\partial_b\eta + \hbar\partial_\Delta\eta)^k(f)$. $\exists$natural diagrammatic interpretation in which $\partial_b\eta$ adds a vertex and $\partial_\Delta\eta$ adds a loop. ev_0 keeps only closed diagrams. Thus get sum of *Feynman diagrams*. `arXiv:1202.1554`

This works more generally for any s.h.BD structure.

8. Quantum field theory and E_d quantization

Suppose have s.h.BD str on $\mathrm{Maps}(M_{\mathrm{dR}}, \mathrm{spec}\,\widehat{\mathrm{Sym}}(V))$. I.e. $\mathrm{Chains}_\bullet(M) \otimes V$ is properadic $\mathbb{D}\,\mathrm{Frob}_0$-alg. Get differential Δ on $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(M) \otimes V)[\![\hbar]\!]$; is quasilocal perturbation Δ of $\partial_0 = \partial_{\mathrm{dR}} \otimes \mathrm{id}_V + \mathrm{id}_{\mathrm{Chains}} \otimes \partial_V$.

HPL: any retraction $\mathrm{H}_\bullet(M) \simeq \mathrm{Chains}_\bullet(M) \Rightarrow$ deformed retraction $\big(\widehat{\mathrm{Sym}}(\mathrm{H}_\bullet(M) \otimes V)[\![\hbar]\!], \tilde\delta\big) \simeq \big(\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(M) \otimes V)[\![\hbar]\!], \Delta\big)$.

Defn: Deformed inclusion is *insertion of observables* along choice of $\mathrm{H}_\bullet \hookrightarrow \mathrm{Chains}_\bullet$. Deformed projection is *expectation value*.

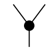
For remainder of talk, $M = \mathbb{R}^d$. $\mathrm{H}_\bullet(\mathbb{R}^d) \otimes V = V$.

Retraction $\mathrm{H}_\bullet(\mathbb{R}^d) \simeq \mathrm{Chains}_\bullet(\mathbb{R}^d)$ is choice of $z \in \mathbb{R}^d$. $\widehat{\mathrm{Sym}}(V) \to \widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)[\![\hbar]\!]$ is *insertion at* z.

Defn: Choose $z_1, \dots, z_n$. Define $\star_{z_1,\dots,z_n} : \widehat{\mathrm{Sym}}(V)^{\otimes n} \to \widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ by: insert $f_i \in \widehat{\mathrm{Sym}}(V)$ at z_i; multiply outputs with commutative product in $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)[\![\hbar]\!]$; take expectation value of product. This is the *n-point function*.

Thm: Suppose all z_i distinct. The *large volume limit* is $\lim_{r\to\infty} \star_{rz_1,\dots,rz_n}$. If $\mathbb{D}\,\mathrm{Frob}_0 \curvearrowright \mathrm{Chains}_\bullet(M) \otimes V$ factors through $\mathrm{QLoc}(M) \otimes \mathrm{End}(V)$, then large volume limit converges in power series topology.

Thm modulo details (I've checked everything when $d = 1$): Large-volume limit of n-point function is n-ary operation in an E_d-algebra structure on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$.

Cor: Suppose have $\mathbb{D}\mathbb{D}\,\mathrm{invFrob}_d \to \mathrm{QLoc}(\mathbb{R}^d)$ sending ,  to Thom forms, with *properadic* bar duals. Earlier abstract nonsense: $\forall\ \mathbb{D}\,\mathrm{Frob}_d$-algebras V, get $\mathbb{D}\,\mathrm{Frob}_0 \to \mathrm{QLoc}(\mathbb{R}^d) \otimes \mathrm{End}(V)$. By above Thm, $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ is E_d alg; we started with s.h.Pois_d-alg $\widehat{\mathrm{Sym}}(V)$.

Calculation: this E_d structure is deformation in the direction of given s.h.Pois_d structure (i.e. get back s.h.Pois_d structure by taking associated graded). Thus *quasilocal homotopy* $\mathrm{invFrob}_d$ *structures on* $\mathrm{Chains}_\bullet(\mathbb{R}^d)$ *give universal* E_d *quantizations*.

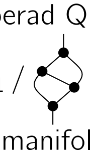
Cor: Above was with completed symmetric algebras. But all formulas restrict also to non-completed symmetric algebras. Any Pois_d alg has resolution as s.h.Pois_d structure on a non-completed symmetric algebra. So get full quantization functor $\{\mathrm{Pois}_d \text{ algebras}\} \to \{E_d \text{ algebras}\}$.

When $d \geq 2$, this is essentially formality of E_d operad.

Remark: $\mathrm{H}_\bullet\,\mathrm{QLoc}(\mathbb{R}^d) = \mathrm{invFrob}_d$. So existence of $\mathbb{D}\mathbb{D}\,\mathrm{invFrob}_d \to \mathrm{QLoc}(\mathbb{R}^d)$ is formality of $\mathrm{QLoc}(\mathbb{R}^d)$.

Conj: $\exists$ canonical homotopy equiv between space of formality morphisms of operad E_d and space of formality morphisms of properad $\mathrm{QLoc}(\mathbb{R}^d)$.

Above outlines one direction. In converse, universal E_d quantization $\Rightarrow$ quantization of $\mathrm{Maps}(M_{\mathrm{dR}}, X) \Rightarrow$ expand in Feynman diagrams and get some operations on $\mathrm{Chains}_\bullet(\mathbb{R}^d)$, which are probably quasilocal homotopy $\mathrm{invFrob}_d$ action.

Thm: Recall that properad $\mathrm{QLoc}(\mathbb{R})$ is not formal. Consider properad $P = \mathrm{LB}_1\,/$ . {Homotopy P-algebras} $\subseteq$ {s.h.Pois_1 infinitesimal manifolds} as those satisfying some equations. For these Poisson manifold, $\exists$ canonical wheel-free quantization, since obstruction theory gives canonical map $\mathbb{D}P \to \mathrm{QLoc}(\mathbb{R})$. **Conj:** This is all Poisson mans with wheel-free quantization.