`arXiv:1409.5934`, w/ Brandenburg and Chirvasitu.

1. Categorified linear algebra

Fix field $\mathbb{K}$ (arbitrary characteristic).

Idea: Colimits, in particular $\sqcup$ or $\oplus$, categorify $+$. Hence:

Defn: Would like bicategory "$\text{COCOMPLETE}_{\mathbb{K}}$" with
- **objects** cocomplete $\mathbb{K}$-linear categories,
- **1-morphisms** cocontinuous $\mathbb{K}$-linear functors,
- **2-morphisms** natural transformations.

But this leads to set-theoretic difficulties: two cocont's functors can have "more than a set" of natural transformations.

So ask for a set-theoretic technical condition, that objects be *locally presentable*:

objects' cocomplete $\mathbb{K}$-linear categories that are generated under colimits by some *set* of objects, such that for each generating object X there is a cardinal κ s.t. $\hom(X, -)$ preserves colimits of diagrams in which every set of $< \kappa$ objects in the diagram has an upper bound in the diagram.

(Second condition says that every object is "set-sized".)

This is the bicategory $\text{PRES}_{\mathbb{K}}$. (**Lemma:** Hom cats in $\text{PRES}_{\mathbb{K}}$ are locally small, so $\text{PRES}_{\mathbb{K}}$ is a bicat.) It includes all examples, and all constructions performed in larger non-bicat $\text{COCOMPLETE}_{\mathbb{K}}$ in fact stay in $\text{PRES}_{\mathbb{K}}$.

Write $\text{HOM}(\mathcal{C}, \mathcal{D})$ for hom cat between $\mathcal{C}, \mathcal{D} \in \text{PRES}_{\mathbb{K}}$.

Remark: Every $F \in \text{HOM}(\mathcal{C}, \mathcal{D})$ has a right adjoint among all functors (not nec. among cocont's functors).

Defn: *Tensor product* of $\mathcal{C}, \mathcal{D} \in \text{PRES}_{\mathbb{K}}$ is $\mathcal{C} \boxtimes \mathcal{D}$ s.t. $\forall \mathcal{E}$, $\text{HOM}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}) = \{$functors $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ s.t. $F(C, -) : \mathcal{D} \to \mathcal{E}$ and $F(-, D) : \mathcal{C} \to \mathcal{E}$ are $\mathbb{K}$-linear and cocont's $\forall C \in \mathcal{C}, D \in \mathcal{D}\}$.

Lemma (Kelly, ...): $\mathcal{C} \boxtimes \mathcal{D}$ exists. $\text{HOM}(\mathcal{C}, \mathcal{D}) \in \text{PRES}_{\mathbb{K}}$. Hom-tensor adjunction.

Unit for $\boxtimes$ is $\text{VECT} = \text{VECT}_{\mathbb{K}}$.

Examples:

- For A, B algs/$\mathbb{K}$, $\text{MOD}_A, \text{MOD}_B \in \text{PRES}_{\mathbb{K}}$. Then $\text{MOD}_A \boxtimes \text{MOD}_B = \text{MOD}_{A \otimes B}$ and $\text{HOM}(\text{MOD}_A, \text{MOD}_B) = {}_A\text{MOD}_B = \{\text{bimodules}\}$. This is "Eilenberg–Watts thm."
- For C a coalgebra/$\mathbb{K}$, $\text{COMOD}^C \in \text{PRES}_{\mathbb{K}}$. But $\boxtimes, \text{HOM}$ are hard to describe: $\nexists$ an "Eilenberg–Watts thm" for coalgebras.
- For X a stack/$\mathbb{K}$, $\text{QCOH}(X) \in \text{PRES}_{\mathbb{K}}$. In general, $\boxtimes, \text{HOM}$ hard to describe, but do have $\text{QCOH}(X) \boxtimes \text{MOD}_A = \text{QCOH}(X \times_{\text{Spec}\,\mathbb{K}} \text{Spec}\, A)$ for A commutative.

Thm (Bird): $\text{PRES}_{\mathbb{K}}$ has all limits and colimits. Limits = limits in CAT. Colimits = limits along right adjoint functors in CAT. Universal property of colimit applies among all cocomplete cats, not just loc. pres. ones.

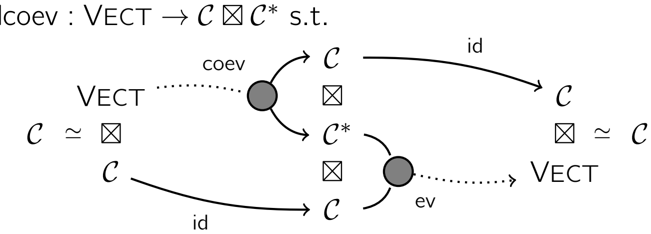
Examples:

- $\text{COMOD}^C = \varinjlim \text{COMOD}^{C_i}$ along diagram of all finite-dim sub-coalgebras $C_i \subseteq C$. This is Sweedler's "fundamental thm of coalgebras." Note: if $\dim C_i < \infty$, $\text{COMOD}^{C_i} = \text{MOD}_{C_i^*}$.
- $\text{QCOH}(X) = \varprojlim \text{QCOH}(\text{Spec}\, A_i)$ for $X = \varinjlim \text{Spec}\, A_i$ an open cover by affines.

2. Dualizability

Defn: Let $\mathcal{C}^* = \text{HOM}(\mathcal{C}, \text{VECT})$. $\mathcal{C}$ is *dualizable* if the canonical map $\mathcal{C}^* \boxtimes \mathcal{C} \to \text{HOM}(\mathcal{C}, \mathcal{C})$ is an equiv.

Lemma: It suffices to check that $\text{id}_{\mathcal{C}}$ is in the essential image of the map. Defn agrees with the usual one in terms of evaluation and coevaluation maps, i.e. that $\exists \text{coev} : \text{VECT} \to \mathcal{C} \boxtimes \mathcal{C}^*$ s.t.



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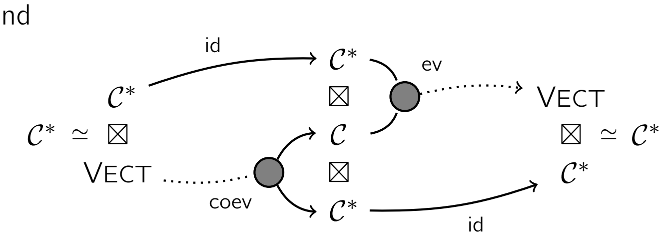


are identities.

Defn: $X \in \mathcal{C}$ is *compact projective* if $\hom(X, -) : \mathcal{C} \to \text{VECT}$ is cocontinuous. (Note: this interprets "projective" as meaning "$\hom(X, -)$ preserves coequalizers." In non-abelian categories, this is stronger than "$\hom(X, -)$ preserves epimorphisms".)

Lemma (Kelly, ...): If $\mathcal{C}$ is locally presentable, its full subcat $\mathcal{C}^{\text{cp}}$ of compact projectives is essentially small.

Defn: $\mathcal{C}$ *has enough compact projectives* if $\mathcal{C}$ is generated by its compact projectives under colimits.

Thm (B–C–JF): If $\mathcal{C} \in \textsc{Pres}_{\mathbb{K}}$ has enough compact projectives, then it is dualizable.

Proof outline: Then $\mathcal{C} = \textsc{Fun}_{\mathbb{K}}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \textsc{Vect})$ and $\mathcal{C}^* = \textsc{Fun}_{\mathbb{K}}(\mathcal{C}^{\mathrm{cp}}, \textsc{Vect})$ and the "generalized Eilenberg–Watts thm":

$$\begin{aligned} &\textsc{Hom}(\textsc{Fun}_{\mathbb{K}}(A, \textsc{Vect}), \textsc{Fun}_{\mathbb{K}}(B, \textsc{Vect})) \\ &\quad = \textsc{Fun}_{\mathbb{K}}(A^{\mathrm{op}} \times B, \textsc{Vect}) \\ &\quad = \textsc{Fun}_{\mathbb{K}}(A^{\mathrm{op}}, \textsc{Vect}) \boxtimes \textsc{Fun}_{\mathbb{K}}(B, \textsc{Vect}). \ \square \end{aligned}$$

Notation advocacy: $\mathcal{C} = \textsc{Fun}_{\mathbb{K}}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \textsc{Vect})$ is a version of "Yoneda embedding" $X \mapsto \hom(-, X)$. In general, Yoneda embedding should always be called "よ", which is first letter of "Yoneda" in Hiragana.

Defn: Affine algebraic group is *linearly reductive* if its representations are completely reducible. *Virtually linearly reductive* if has lin. red. finite-index subgroup.

Examples:

- Finite G in characteristic dividing $|G|$ is virtually linearly reductive but not linearly reductive.
- $\mathbb{G}_a$ is not virtually linearly reductive.

Cor of thm: X an affine scheme acted on by G a vir. lin. red. group. Then $\textsc{QCoh}([X/G])$ is dualizable.

Proof outline: Donkin: G is vir. lin. red. iff $\textsc{Rep}(G)$ has enough cpt proj's. Serre: X is affine iff $\textsc{QCoh}(X)$ has enough cpt proj's. Some unpacking $\Rightarrow$ $\textsc{QCoh}([X/G])$ has enough cpt proj's. $\square$

Remark: Dualizability survives field extensions.

Remark (Donkin): TFAE: (i) G is vir. lin. red. (ii) $\textsc{Rep}(G)$ has *any* non-zero projective. (iii) *Every* injective is projective.

3. Nondualizability

Thm (B–C–JF): For C a coalgebra, $\textsc{Comod}^C$ is dualizable iff it has enough projectives.

Proof outline: ($\Leftarrow$) $\textsc{Comod}^C$ has enough proj's iff it has enough cpt proj's. ($\Rightarrow$) The fun part:

- Recall $\textsc{Comod}^C = \varinjlim \textsc{Comod}^{C_i}$ for C_i the finite-dim subcoalgs of C. Then $(\textsc{Comod}^C)^* \boxtimes \textsc{Comod}^C = \varinjlim((\textsc{Comod}^C)^* \boxtimes \textsc{Comod}^C)$, so its essential image in $\textsc{End}(\textsc{Comod}^C)$ consists of all endofunctors of form $F = \varinjlim F_i$ s.t. F_i factors through $\textsc{Comod}^{C_i}$.
- Lin: If $\textsc{Comod}^C$ does not have enough projectives, then it has a simple S s.t. $\forall i$, $\exists$ essential projection $T_i \twoheadrightarrow S$ such that $T_i \notin \textsc{Comod}^{C_i}$.
- Suffice to show $\forall F$, $\nexists \theta : F \overset{\sim}{\Rightarrow} \mathrm{id}$. Since $F = \varinjlim F_i$ and $S \neq 0$, suffice to show $\forall i, \theta_i = 0 : F_i(S) \to S$.

Naturality and cocontinuity: $\begin{array}{ccc} F_i(T_i) & \xrightarrow{\theta_i(T_i)} & T_i \\ \downarrow & & \downarrow \\ F_i(S) & \xrightarrow{\theta_i(S)} & S \end{array}$.

$\theta_i(T_i)$ cannot be onto, since $T_i \notin \textsc{Comod}^{C_i}$. Since $T_i \twoheadrightarrow S$ is essential, $F(T_i) \to S$ is 0. $\square$

Thm (B–C–JF): Nevertheless, $((\textsc{Comod}^C)^*)^* = \textsc{Comod}^C$ if $\dim C \leq \aleph_0$. Unknown for $\dim C > \aleph_0$.

Thm (B–C–JF): Suppose X a scheme contains closed dim>0 subscheme Y s.t. $f : Y \hookrightarrow \mathbb{P}^N$ closed. Then $\textsc{QCoh}(X)$ is not dualizable.

Proof outline:

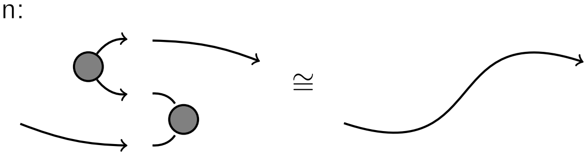
- If $\textsc{QCoh}(X)$ is dualizable, so is $\textsc{QCoh}(Y)$ for any closed subscheme $Y \subseteq X$.
- Similar proof as above, using $\bigoplus f^*\mathcal{O}(n)$s. $\square$

Conj: $\textsc{QCoh}(X)$ dualizable iff X affine.
Stronger conj: $\mathcal{C}$ dualizable iff has enough cpt proj's.

4. Comments on quantum field theory

A *categorified quantum field theory* uses categories in place of Hilbert spaces. Categorified qfts come from extended qfts (as codim-2 part) and from relative qfts (as twists/anomalies).

Dualizability ev/coev zig-zag = Morse handle cancelation:



Hence categorified *topological* qfts only use dualizable categories.

$\textsc{Rep}(G) \leftrightarrow$ pure (topological) gauge theory.

$\textsc{QCoh}(X) \leftrightarrow$ (topological) sigma model with target X.

$\textsc{QCoh}([X/G]) \leftrightarrow$ gauge theory with matter.

Our nondualizability results $\Rightarrow$ such theories can only be described by $\textsc{Pres}_{\mathbb{K}}$ when G is virtually linearly reductive and X does not contain a projective subvariety.

OTOH, Ben-Zvi–Francis–Nadler: the *derived* categories corresponding to all these examples are dualizable .

Moral: Quantum field theory requires derived categories.