Talk based on `arXiv:1507.06297`.

0. Talk punchline

$\mathbb{C}$	Orientations	Hermiticity
SUPERVECT	Spin structures	Spin-Statistics

1. Local structures on manifolds

Let $\mathrm{MAN}_d = \{d\text{-manifolds and local diffeomorphisms}\}$. (Topologized $\rightsquigarrow$ $(\infty,1)$-category.) Let $\mathcal{X}$ an ∞-topos.

Defn: *Topological local structure* = sheaf $\mathcal{G} : \mathrm{MAN}_d \to \mathcal{X}$.

Non-e.g.: $M \mapsto \{\text{metrics on } M\}$ is a sheaf on the strict category $\mathrm{MAN}_d^{\text{strict}}$ but not on the $(\infty,1)$-category.

Lemma (“Cobordism Hypothesis”): Topological local structures valued in $\mathcal{X}$ are classified by $\mathcal{X}^{\mathrm{O}(d)} = \{\mathcal{X}\text{-objects equipped with } \mathrm{O}(d)\text{-action}\}$.

Pf: Use $\mathrm{O}(d) \simeq \hom_{\mathrm{MAN}_d}(\mathbb{R}^d, \mathbb{R}^d)$ together with existence of good open covers. □

E.g.: Given $\mathrm{O}(d) \curvearrowright X$ in $\mathcal{X}^{\mathrm{O}(d)}$, corresponding sheaf is $M \mapsto \mathrm{maps}_{\mathrm{O}(d)}(\mathrm{Fr}(M), X)$.

- Trivial action $\rightsquigarrow$ *topological sigma-model*.
- $\{G\text{-tangential structures}\} \leftrightsquigarrow X = \mathrm{O}(d)/G$.
- $\{\text{orientations}\} \leftrightsquigarrow X = \pi_0(\mathrm{O}(\infty)) = \mathbb{Z}/2$.
- $\{\text{spin strs}\} \leftrightsquigarrow X = \pi_{\leq 1}(\mathrm{O}(\infty)) = \mathbb{Z}/2 \times \mathrm{B}(\mathbb{Z}/2)$.

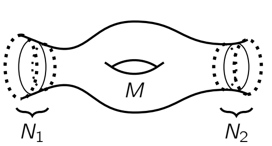
Why allow other topoi $\mathcal{X}$? So that local structures can vary in moduli. If G is algebraic group, then $\mathrm{Loc}_G(-)$ is valued in algebraic stacks. If G is super Lie group, then $\{G\text{-bundles with connection}\}$ is valued in topos of sheaves on site of supermanifolds (and smooth maps).

2. Locally structured bordism category

Let $\mathrm{BORD}_d = \mathrm{BORD}_d^{\text{smooth}}$ [Calaque–Scheimbauer, Lurie]. I.e. k-morphisms are k-dim smooth cobordisms between $(k-1)$-morphisms, with no extra local data.

Let $\mathrm{SPANS}_d(\mathcal{X})$ be sym mon (∞,d)-cat with k-morphisms $= k$-fold spans in $\mathcal{X}$, $\circ$ = fiber product [Haugseng].

Given $\mathcal{G} : \mathrm{MAN}_d \to \mathcal{X}$, can assign to each bordism in BORD_d a span in $\mathcal{X}$:



$$\overset{\mathcal{G}}{\mapsto} \quad \begin{array}{ccc} & \{\mathcal{G}\text{-structures on } M\} & \\ {}^{\text{restrict}}\swarrow & & \searrow^{\text{restrict}} \\ \{\mathcal{G}\text{-strs on } N_1\} & & \{\mathcal{G}\text{-strs on } N_2\} \end{array}$$

This defines a sym mon functor $\mathcal{G} : \mathrm{BORD}_d \to \mathrm{SPANS}_d(\mathcal{X})$. Sheaf condition $\Rightarrow$ composition. Topological (i.e. use $(\infty,1)$-cat MAN_d, not just $\mathrm{MAN}_d^{\text{strict}}$) $\Rightarrow$ units.

Prop [Li-Bland]: When $\mathcal{X}$ is ∞-topos, $\mathrm{SPANS}_d(\mathcal{X})$ is an (∞,d)-category *internal to* $\mathcal{X}$, i.e. k-morphisms can vary in $\mathcal{X}$-parameterized moduli. □

Defn: Treat BORD_d as $\mathcal{X}$-internal category via unique geometric morphism $\mathrm{SPACES} \leftrightarrows \mathcal{X}$. Let $\mathcal{X}_{\{\mathrm{pt}\}/} = \{\text{pointed } \mathcal{X}\text{-objects}\}$. ($\{\mathrm{pt}\} \in \mathcal{X}$ is initial object.) The *$\mathcal{G}$-structured bordism category* is the pullback of $\mathcal{X}$-internal categories

$$\begin{array}{ccc} \mathrm{BORD}_d^{\mathcal{G}} & \longrightarrow & \mathrm{SPANS}_d(\mathcal{X}_{\{\mathrm{pt}\}/}) \\ \downarrow & \ulcorner & \downarrow {\scriptstyle \text{Forget the pointing}} \\ \mathrm{BORD}_d & \xrightarrow{\mathcal{G}} & \mathrm{SPANS}_d(\mathcal{X}) \end{array}$$

(Internal $\rightsquigarrow$ local structure can vary in moduli.)

Defn: Let $\mathcal{V}$ be $\mathcal{X}$-internal sym mon (∞,d)-category. A *$\mathcal{V}$-valued $\mathcal{G}$-structured field theory* is functor of $\mathcal{X}$-internal sym mon (∞,d)-categories $\mathrm{BORD}_d^{\mathcal{G}} \to \mathcal{V}$.

Lemma: Let $\mathrm{SPANS}_d(\mathcal{X};\mathcal{V}) = (\infty,d)$-category whose k-morphisms are k-fold spans in $\mathcal{X}$ equipped with bundles of k-morphisms in $\mathcal{V}$ [Haugseng]. $\mathcal{V}$-valued $\mathcal{G}$-structured field theory = lift of (non-internal!) functors:

$$\begin{array}{ccc} & & \mathrm{SPANS}_d(\mathcal{X};\mathcal{V}) \\ & \nearrow & \downarrow {\scriptstyle \text{Forget the } \mathcal{V}\text{-bundle}} \\ \mathrm{BORD}_d & \xrightarrow{\mathcal{G}} & \mathrm{SPANS}_d(\mathcal{X}) \end{array}$$

Pf: Unpack some adjunctions. □

Question: Explain “qfts fibered over $\mathcal{G}$” as “$\mathcal{G}$-$\mathcal{V}$-twisted” field theories a la Stolz–Teichner.

3. Hermitian field theory

In SPACES, sheaf $\mathrm{Or} = \{\text{orientations}\}$ is classified by unique $\mathbb{Z}/2$-torsor. Let $\mathcal{X} = \mathrm{STACKS}_{\mathbb{R}}$. Then $\mathrm{Spec}(\mathbb{C})$ is *another* $\mathbb{Z}/2$-torsor, and is the unique nontrivial one.

Defn: Sheaf $\mathrm{Her} : \mathrm{MAN}_d \to \mathrm{STACKS}_{\mathbb{R}}$ of *Hermitian structures* is classified by $\mathrm{Spec}(\mathbb{C}) \in (\mathrm{STACKS}_{\mathbb{R}})^{\mathrm{O}(d)}$.

Lemma: $\mathrm{Her}(-) = \dfrac{\mathrm{Or}(-) \times \mathrm{Spec}(\mathbb{C})}{\mathbb{Z}/2}$. □

Recall stack of categories (i.e. $\mathrm{STACKS}_{\mathbb{R}}$-internal) category $\mathrm{QCOH} : A \mapsto \mathrm{MOD}_A$. Normal to demand that for *quantum field theory*, d-manifolds $\mapsto$ numbers, $(d-1)$-manifolds $\mapsto$ vector spaces. In our case, this becomes:

$$\Omega^{d-1}\mathcal{V} = \mathrm{QCOH}.$$

(Sym mon (∞,k)-cat $\mathcal{C} \rightsquigarrow$ sym mon $(\infty,k-1)$-cat $\Omega\mathcal{C} = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ called its *looping*. **E.g.:** $\Omega\mathrm{QCOH} = \mathcal{O}$.)

E.g.: Let $Z : \mathrm{BORD}_d^{\mathrm{Her}} \to \mathcal{V}$.

• Suppose M^d is connected orientable. Then $\mathrm{Her}(M) \cong \mathrm{Spec}(\mathbb{C})$, but *not canonically*. $Z(M) \in \mathcal{O}(\mathrm{Her}(M))$. Each orientation of $M \rightsquigarrow Z(M) \in \mathbb{C}$; orientation reversal = complex conjugation.

• Suppose N^{d-1} is connected orientable. Each orientation gives iso $\mathrm{Her}(N) \cong \mathrm{Spec}(\mathbb{C})$, hence $Z(N) \in \mathrm{QCOH}(\mathrm{Spec}(\mathbb{C})) = \mathrm{VECT}_{\mathbb{C}}$. $Z(N \times ⅄)$ is a nondegenerate symmetric *sesquilinear* form on $Z(M)$. (Not necessarily positive definite.) Hence name “Hermitian.”

Punchline: “Oriented” and “Hermitian” are the two versions of “étale-locally-over-$\mathbb{R}$ oriented.”

4. Categorified torsors

Why does the nontrivial $\mathbb{Z}/2$-torsor over $\mathbb{R}$ exist?

0. $\mathbb{C}$ is non-zero finite dim com $\mathbb{R}$-algebra.
1. (Field) Any non-zero map $\mathbb{C} \to A$ of finite dim com $\mathbb{R}$-algebras is injective.
2. (Algebraically closed) Any non-zero finite-dim com $\mathbb{R}$-algebra A admits $A \to \mathbb{C}$.
3. (Galois) $\mathrm{MOD}_{\mathbb{R}} = \mathrm{MOD}_{\mathbb{C} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})}$.

Then *G-torsors* (i.e. $G \curvearrowright T \to \mathrm{Spec}(\mathbb{R})$ s.t. $T/G \overset{\sim}{\to} \mathrm{Spec}(\mathbb{R})$ and $T \times G \overset{\sim}{\to} T \times_{\mathrm{Spec}(\mathbb{R})} T$) are classified by $\mathrm{maps}(\mathrm{B}\,\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{B}G)$.

Defn: A *categorified com $\mathbb{R}$-algebra* is a (nice!) $\mathbb{R}$-linear sym mon cat. **E.g.:** {com $\mathbb{R}$-algs} $\hookrightarrow$ {cat com $\mathbb{R}$-algs} via $A \mapsto (\mathrm{MOD}_A, \otimes_A)$. **Defn:** $\mathrm{CATSTACKS}_{\mathbb{R}}$ = stacks on site of cat com $\mathbb{R}$-algs. **E.g.:** $\mathrm{QCOH} : \mathcal{C} \mapsto \mathcal{C}$.

Defn: Cat com $\mathbb{R}$-alg $(\mathcal{C}, \otimes, \dots)$ is *finite dim* if (a) $\mathcal{C} \simeq \mathrm{MOD}_A$ for finite dim *associative* alg A (so that underlying cat is finite-dim), and (b) projective $\Rightarrow$ dualizable (so that “internal” and “external” notions of “finite” agree).

E.g.: $\mathrm{SUPERVECT}_{\mathbb{C}}$, as *non-sym* monoidal category, is $\mathrm{REP}_{\mathbb{C}}(\mathbb{Z}/2)$. Its braiding is determined by $\sigma = -1 : \Pi \otimes \Pi \to \Pi \otimes \Pi$. ($\Pi$ = sign rep of $\mathbb{Z}/2$.)

Thm: I. The categorified algebraic closure of $\mathbb{R} \equiv \mathrm{VECT}_{\mathbb{R}}$ is $\mathrm{SUPERVECT}_{\mathbb{C}}$. **II.** The extension is Galois with Galois group $\mathrm{Gal}(\mathrm{SUPERVECT}_{\mathbb{C}}/\mathbb{R}) = \mathbb{Z}/2 \times \mathrm{B}(\mathbb{Z}/2)$.

Pf: I. Deligne’s “existence of super fiber functors” plus small modifications. **II.** The $\mathbb{Z}/2$ acts by complex conjugation. The $\mathrm{B}(\mathbb{Z}/2)$ acts by “$(-1)^f$” = natural auto of identity s.t. $(-1)^f|_{\mathbb{1}} = +1$ and $(-1)^f|_{\Pi} = -1$. □

Exercise: $\mathrm{VECT}_{\mathbb{R}}$ has five field extensions: $\mathrm{VECT}_{\mathbb{R}}$, $\mathrm{VECT}_{\mathbb{C}}$, $\mathrm{SUPERVECT}_{\mathbb{R}}$, $\mathrm{SUPERVECT}_{\mathbb{C}}$, and “$\mathrm{SUPERVECT}_{\mathbb{H}}$”, which is $\mathrm{VECT}_{\mathbb{R}} \boxplus \mathrm{MOD}_{\mathbb{H}}$ with interesting sym mon str.

5. Spin–statistics field theories

Take $\mathcal{X} = \mathrm{CATSTACKS}_{\mathbb{R}}$. There is a canonical nontrivial $\pi_{\leq 1}\mathrm{O}(\infty) = \mathbb{Z}/2 \times \mathrm{B}(\mathbb{Z}/2)$ torsor, namely Galois action on $\mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{R}})$. Just as in Hermitian case, corresponding local structure $\mathrm{SpinStat} : \mathrm{MAN}_d \to \mathcal{X}$ is *(categorified) étale-locally* equivalent to Spin = {spin structures}. Moreover, just as in Hermitian case:

$$\mathrm{SpinStat}(-) = \frac{\mathrm{Spin}(-) \times \mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{C}})}{\mathbb{Z}/2 \times \mathrm{B}(\mathbb{Z}/2)}$$

How does $\mathbb{Z}/2 \times \mathrm{B}(\mathbb{Z}/2)$ act on $\mathrm{Spin}(-)$? The $\mathbb{Z}/2$ part acts by orientation reversal. The $\mathrm{B}(\mathbb{Z}/2)$ part acts by the nontrivial Spin automorphism of a spin manifold, called “$\looparrowright$” or “rotate by 360°”. Let N be Spin; then the mapping cylinder of N is the Spin manifold $N \times [0,1]$ with Spin structure $N \times \looparrowright$. N.b.: $\looparrowright\looparrowright = \frown$ (belt trick).

Suppose $\dim N = d-1$ and Z is a SpinStat field theory. Then $Z(N) \in \mathrm{QCOH}(\mathrm{SpinStat}(N))$ = disjoint union of $\mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{C}})$s, one for each $(\mathbb{Z}/2 \times \mathrm{B}(\mathbb{Z}/2))$-orbit in $\mathrm{Spin}(N)$. So for each spin structure, $Z(N)$ gives a complex supervector space. Compatibility with $\mathbb{Z}/2$-action $\Rightarrow$ Hermitian supervector space.

Defn: In a field theory valued in supervector spaces, a *fermion* is a (-1)-eigenstate of $(-1)^f$. In a spin field theory, a *spinor* is a (-1)-eigenstate of $\looparrowright$.

In a SpinStat field theory, $\looparrowright = (-1)^f$, i.e. they are *spin-statistics* in the sense that fermions=spinors.

6. A topological spin-statistics theorem

There are spin theories that are not spin-statistics, and super field theories that are not spin. However:

Thm [TJF]: If Z is étale-locally-spin TFT over $\mathbb{R}$ such that the unoriented theory $\int_{\text{spin structures}} Z$ is positive, then Z is Hermitian and spin-statistics.

(Hermitian + positive = *unitary*.)

7. Conjectures about categorified Galois groups

Conj [TJF]: Under further categorification, the *infinitely-categorified* absolute Galois group of $\mathbb{R}$ is $\mathrm{O}(\infty)$.

Conj [Ostrick]: If $p \geq 3$, the cat alg closure of $\bar{\mathbb{F}}_p$ is the *Verlinde category* VER_p, a mod-p version of SU(2)-at-level-$(p-2)$. **Conj [Etingof]:** Weirder $p = 2$ case.

Rmk: Extension $\mathrm{VECT}_p \to \mathrm{VER}_p$ is not separable. Maximal separable subextension is $\mathrm{VECT}_p \to \mathrm{SUPERVECT}_p$. So SUPERVECT is still the universal torsor.