Details at `arXiv:1307.5812`. dg, char=0.

1. The BV formalism

Defn: *Classical Field Theory* = the study of those PDE determined by "least action" variational principles = geometry of critical loci in Maps(spacetime, target).

(N.B. target is usually stacky; these days derived, too.)

Defn: *QFT* = computing $\int$(observable) $\exp\left(\frac{i}{\hbar}(\text{action})\right)$, with domain of integration Maps(spacetime, target).

Classical BV formalism: Derived critical locus of any function has symplectic form of hom degree $+1$, i.e. Poisson bracket of deg -1 (convention: $\deg(\partial) = -1$).

BV say: *Any* dg manifold with deg-(-1) Poisson bracket should be considered as a critical locus.

Quantum BV formalism: Twisted de Rham complex for oscillating measure $\exp\left(\frac{i}{\hbar}(\text{action})\right)$ is graded com alg, with $\hbar$-dependent second-order diff. op. Δ s.t. (i) Δ is differential, (ii) $\Delta(1) = 0$, (iii) $\Delta|_{\hbar=0}$ is derivation.

BV say: *Any* graded manifold with such Δ should be considered as an oscillating integral problem.

Historical aside: Batalin–Vilkovisky were physicists, using only $\mathbb{Z}/2$ ("super") gradings. What's called a "BV algebra" in mathematics is not what B–V discovered. It is (almost) the same with $\mathbb{Z}/2$ gradings, but different with $\mathbb{Z}$ gradings. Costello–Gwilliam name what B–V used "BD algebra," after Beilinson–Drinfeld, who used correct gradings in book on CFT.

Polemical aside: Actual derived critical loci / twisted de Rham complexes are always *cotangent bundles*. Why not work with those? Because of dualities / symmetries / gauge equivalence. Usual BV keeps requirement that bracket be *symplectic*, i.e. nondegenerate.

But *symplectic is wrong*. Locally, Poisson = family of symplectics, and working in families is important. Globally, can have rich dualities/etc., so "families of symplectics" isn't good enough: need Poisson.

Defn: *Semistrict homotopy Pois$_d$ structure* on graded com alg A is system of multiderivations making $A[1-d]$ into L_∞ alg. "Semistrict" = don't weaken Leibniz.

Defn: *s.h.BD structure* is differential Δ on $A[\![\hbar]\!]$, $\Delta(1) = 0$, such that $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$ is $(n+1)$th order diff. op.

Exercise: Principal symbols of $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$ are together a s.h.Pois$_0$ structure. **Defn:** Finding Δ for prescribed s.h.Pois$_0$ structure is *(perturbative) quantization*.

Challenge: Find interesting s.h.Pois$_0$/s.h.BD structures on mapping spaces. Interpret as classical/quantum FT.

Thm (Alexandrov–Kontsevich–Schwarz–Zaboronsky): M is closed oriented d-dim manifold. X is *symplectic* Pois$_d$. Then Maps(M_{dR}, X) = derived space of locally constant maps $M \to X$ is symplectic Pois$_0$.

With one lie. It is symplectic: has 2-form with trivial kernel. But ∞-dim, so how to invert to Poisson structure? Also earlier polemic. And what about open M?

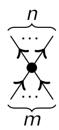
2. Infinitesimal manifolds

I have an answer when X = infinitesimal manifold.

Defn: An *infinitesimal manifold (with local coord chart)* is $\operatorname{Spec}\widehat{\operatorname{Sym}}(V)$ = "formal nbhd of $0 \in V^*$" for a chain complex V. ($\widehat{\operatorname{Sym}}$ = completed symmetric algebra. All geometry should be cont's for power series topology.)

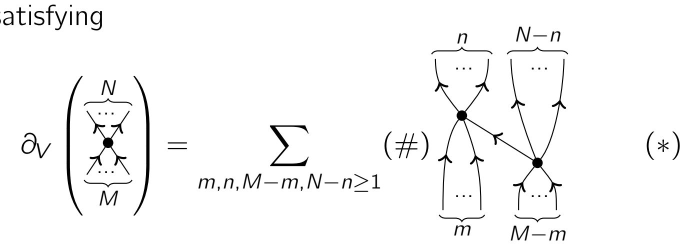
Technical convenience: Assume everything vanishes at $0 \in \operatorname{Spec}\widehat{\operatorname{Sym}}(V)$; absorb linear terms into ∂_V.

Exercise: A s.h.Pois$_d$ structure on $\operatorname{Spec}\widehat{\operatorname{Sym}}(V)$ is same as system of tensors



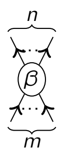
$: V^{\otimes m} \to V^{\otimes n}$ of hom degree $d(m-1)-1$

in $(\text{sign})^{\otimes d} \otimes (\text{triv})$ subrep of $\mathbb{S}_m \otimes \mathbb{S}_n \curvearrowright \hom(V^{\otimes m}, V^{\otimes n})$, satisfying



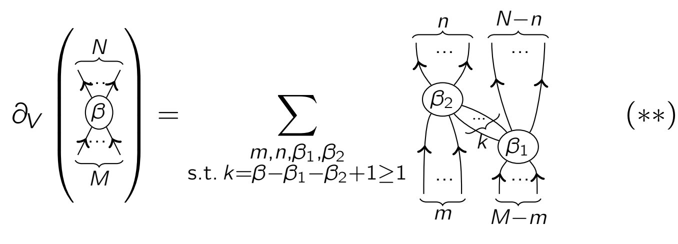
Coeffs (#) depend on conventions. Average over permutations of input/output strands, with signs when d = odd.

Exercise: A s.h.BD structure on $\operatorname{Spec}\widehat{\operatorname{Sym}}(V)$ is system of $\mathbb{S}_m^{\mathrm{op}} \otimes \mathbb{S}_n$-invariant degree-$(-1)$ tensors



$: V^{\otimes m} \to V^{\otimes n}$, labeled by $\beta \in \mathbb{N}$

satisfying



β is *internal genus*. Eqn (∗∗) is homogeneous for *total genus* = internal genus + genus of diagram.

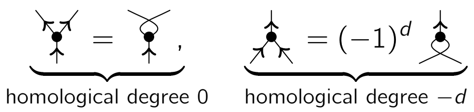
3. Dioperads and properads

Defn: *Associative algebras* have compositions for each arrangement of beads on a string. Similarly:

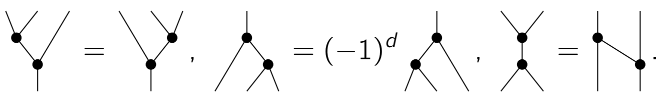
- E_d *algebras* $\leftrightarrow$ beads on $\mathbb{R}^d$
- *operads* $\leftrightarrow$ rooted trees
- *dioperads* $\leftrightarrow$ directed trees
- *properads* $\leftrightarrow$ connected acyclic directed graphs
- *props* $\leftrightarrow$ acyclic directed graphs

E.g.: V a chain complex $\rightsquigarrow$ dioperad/properad/prop $\mathrm{End}(V)(m,n) = \hom(V^{\otimes m}, V^{\otimes n})$. An *action* of P on V (equiv, V is a *P-algebra*) is a morphism $P \to \mathrm{End}(V)$.

E.g.: Dioperad/properad/prop Frob_d of *open d-shifted commutative Frobenius algebras* has generators:



and relations:



Involutive Frob_d: impose $= 0$. (Automatic if d odd.)

Thm (Vallette et al): {dioperads}, {properads}, ..., are model categories with fibration = surjection and acyclic = quasiisomorphic.

Defn: A *homotopy action* of P is an action of any cofibrant replacement $\mathrm{h}P$ (different cofibrant replacements give canonically homotopy equiv notions).

Warning: Free : {properads} $\to$ {props} is exact, but Free : {dioperads} $\to$ {props} is not exact. So propic and properadic notions of "homotopy P-algebra" are the same, but dioperadic notion is generally different.

Lots of technology (Gan, Vallette): Dioperads & properads have good notions of "quadratic" and "Koszul" and "(co)bar construction." (Props don't.) Denote cobar dual of P by $\mathbb{D}P$ = free thing gen by $P^*[-1]$ with $\partial \leftrightarrow$compositions; use $\mathbb{D}^{\mathrm{di}}P$ or $\mathbb{D}^{\mathrm{pr}}P$ if ambiguous.

$\mathrm{h}P = \mathbb{D}\mathbb{D}P \twoheadrightarrow P$ is always a cofibrant replacement.

E.g.: Dioperad Frob_d and properad $\mathrm{invFrob}_d$ are Koszul; Koszul dual LB_d controls Lie bialgs with deg(bracket) = $d-1$ and deg(cobracket) = -1. (Usual LB $\leftrightarrow$ LB_2.)

E.g.: Eqn $(*)$ describes $\mathbb{D}^{\mathrm{di}}\,\mathrm{Frob}_d = \mathbb{D}^{\mathrm{pr}}\,\mathrm{invFrob}_d$. **Cor:** $\mathrm{Spec}\,\widehat{\mathrm{Sym}}(V)$ is s.h.Pois_d iff V is homotopy LB_d.

E.g.: Eqn $(**)$ describes $\mathbb{D}^{\mathrm{pr}}\,\mathrm{Frob}_0$. **Warning:** Koszulity of Frob_0 is not known, so don't know $\mathrm{H}_\bullet(\mathbb{D}^{\mathrm{pr}}\,\mathrm{Frob}_0)$.

Thm (abstract nonsense): There are canonical "sum-over-diagrams" maps $\mathbb{D}\,\mathrm{Frob}_0 \to \mathrm{h}P \otimes \mathbb{D}P$ for any P.

4. Homotopy chain-level Frobenius structures

Recall: want $\mathrm{Maps}(M_{\mathrm{dR}}, \mathrm{Spec}\,\widehat{\mathrm{Sym}}(V))$ to be s.h.Pois_0 or s.h.BD. What is this space? $\mathrm{Spec}\,\widehat{\mathrm{Sym}}(V) \approx V^*$

$\Rightarrow$ $\mathrm{Maps}(M_{\mathrm{dR}}, V^*) \approx \mathcal{O}(M_{\mathrm{dR}}) \otimes V^* \approx \Omega^\bullet_{\mathrm{dR}}(M) \otimes V^*$

$\Rightarrow$ $\mathrm{Maps}(M_{\mathrm{dR}}, V^*) \approx \mathrm{Spec}\,\widehat{\mathrm{Sym}}\big(\mathrm{Maps}(M_{\mathrm{dR}}, V^*)^*\big)$

$\approx \mathrm{Spec}\,\widehat{\mathrm{Sym}}\big((\Omega^\bullet_{\mathrm{dR}}(M) \otimes V^*)^*\big)$

$\approx \mathrm{Spec}\,\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(M) \otimes V\big)$. Take as defn.

Cor: We win the challenge if V is $\mathbb{D}P$-alg and $\mathrm{Chains}_\bullet(M)$ is $\mathrm{h}P$-alg, as then $\mathrm{Chains}_\bullet(M) \otimes V$ is $(\mathrm{h}P \otimes \mathbb{D}P)$-alg hence $\mathbb{D}\,\mathrm{Frob}_0$-alg. Dioperad vs. properad is classical vs. quantum is s.h.Pois_0 vs. s.h.BD.

New challenge: $\mathrm{H}_\bullet(M)$ is Frob_d. Is $\mathrm{Chains}_\bullet(M)$ $\mathrm{h}\,\mathrm{Frob}_d$?

Wrong answer: $\mathrm{H}_\bullet \simeq \mathrm{Chains}_\bullet$, so use homotopy transfer theory to build $\mathrm{h}\,\mathrm{Frob}_d$ action on $\mathrm{Chains}_\bullet(M)$. Why wrong? Result is highly nonlocal. But true locality (all operations supported on diagonal) is too strong: must perturb slightly for good intersection theory.

Defn: A *quasilocal action* is family of actions s.t. all operations have proper support converging to diagonal.

Thm: $\forall M$ oriented d-dim, there is homotopically-unique quasilocal $\mathrm{h}^{\mathrm{di}}\,\mathrm{Frob}_d$ action on $\mathrm{Chains}_\bullet(M)$.

Defn: Get the *classical Poisson AKSZ construction*.

Thm: In dimension 1, quasilocal $\mathrm{h}^{\mathrm{pr}}\,\mathrm{Frob}_1$ structures do not exist! **Pf:** Obstructed in genus 2. `arXiv:1308.3423`.

5. On quantization

Focus on $M = \mathbb{R}^d$. $\mathrm{H}_\bullet$ is $\mathrm{invFrob}_d$, so hope $\mathrm{Chains}_\bullet(\mathbb{R}^d)$ is $\mathrm{h}^{\mathrm{pr}}\,\mathrm{invFrob}_d$, so that V can be $\mathbb{D}^{\mathrm{pr}}\,\mathrm{invFrob}_d = \mathrm{s.h.Pois}_d$.

Each $\vec{z} \in \mathbb{R}^d$ determines deformation retraction

$$\mathbb{Q} \underset{\iota_{\vec z}}{\overset{p_{\vec z}}{\rightleftarrows}} \big(\mathrm{Chains}_\bullet(\mathbb{R}^d), \partial\big) \circlearrowright \eta_{\vec z}\,, \qquad [\partial,\eta] = \mathrm{id} - \iota p$$

Homological perturbation lemma $\Rightarrow$ any s.h.BD str on $\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V$ gives perturbed deformation retraction

$$\big(\widehat{\mathrm{Sym}}(V)[\![\hbar]\!], \tilde\delta\big) \underset{\tilde\iota_{\vec z}}{\overset{\tilde p_{\vec z}}{\rightleftarrows}} \big(\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V\big)[\![\hbar]\!], \partial + \Delta\big)$$

Defn: *n-point function* $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]^{\otimes n} \to \widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$, depending on $\vec z_1, \dots, \vec z_n \in \mathbb{R}^d$, is

$$f_1 \otimes \dots \otimes f_n \mapsto \tilde p_0\big(\iota_{\vec z_1}(f_1) \odot \dots \odot \iota_{\vec z_n}(f_n)\big)$$

where $\odot$ is com product in $\widehat{\mathrm{Sym}}$.

Thm (modulo checking some things): Given quasilocal $\mathrm{h}\,\mathrm{invFrob}_d$ structure on $\mathrm{Chains}_\bullet(\mathbb{R}^d)$, and s.h.$\mathrm{Pois}_d$ target V, the n-point functions are together an E_d alg structure on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$. So {quasilocal $\mathrm{h}\,\mathrm{invFrob}_d$ structures on $\mathrm{Chains}_\bullet(\mathbb{R}^d)$} = {$E_d$ universal quantizations} (when $d \geq 2$, universal quantization = formality).