See `arXiv:1412.4664` and `arXiv:1307.5812`.

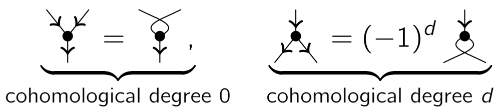
1. Global to local

"Poincaré duality" is a big thing. Here is a small piece:

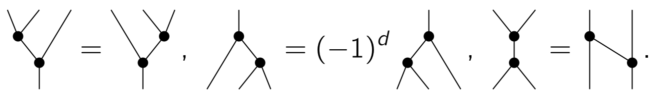
Basic fact: Let M be compact oriented manifold. Then its real de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is a "shifted" commutative Frobenius algebra (over $\mathbb{R}$).

"Shifted": trace has non-zero cohomological degree.

Defn: An *open and coopen d-shifted commutative Frobenius algebra* is a $\mathbb{Z}$-graded (or dg) v-space with maps



such that



I will call this a "Frob_d-algebra." Non-commutative Frob algebras will not appear. Opposite of "open" is "unital." Opposite of "coopen" is "counital." $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is unital and counital, but I will not use this.

Remark: When Euler characteristic $\chi(M) = 0$, then additionally $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ satisfies  $= 0$. Such structure is *involutive Frobenius*, called $\mathrm{invFrob}_d$. (When d odd, signs $\Rightarrow$  $= 0$, i.e. $\mathrm{Frob}_d = \mathrm{invFrob}_d$.)

Motivating question: Does this structure come from something "local" on M? If so, it is a piece of "local Poincaré duality."

Refined question: Do these maps lift to "local" operations on $\Omega^\bullet_{\mathrm{dR}}(M)$, making $\Omega^\bullet_{\mathrm{dR}}(M)$ into a "derived" or "homotopy" Frob_d-algebra?

Why you might want such structure: "Local" version of product on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is wedge product, making $\Omega^\bullet_{\mathrm{dR}}(M)$ into dgca. Homotopy transfer theory along $\Omega^\bullet_{\mathrm{dR}}(M) \simeq \mathrm{H}^\bullet_{\mathrm{dR}}(M)$ produces "homotopy commutative" algebra on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ with generally-nontrivial associators (the *Massey products*), which encode topological info about M. Maybe same works for Frobenius?

Exercise: $\Omega^\bullet_{\mathrm{dR}}(M)$ does not admit *strict* Frob_d-structure.
Hint: Partitions of unity.

2. Dioperads and properads

To pose the question precisely, we need to answer first:

Pre-question: What is a "homotopy Frob_d algebra"?

If Frob_d were an operad, an answer would be "an action of Frob_d in the $(\infty, 1)$-category of operads," i.e. "an action of a cofibrant replacement of Frob_d."

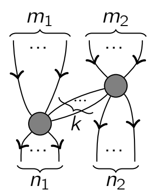
But Frob_d involves many-to-many operations, so "operad" is the wrong framework.

Defn by analogy:

algebraic gadget	diagrammatics
operads	rooted trees
dioperads	directed trees
properads	directed connected graphs with no directed cycles
props	directed graphs with no directed cycles
wheeled properads	directed connected graphs
wheeled props	directed graphs

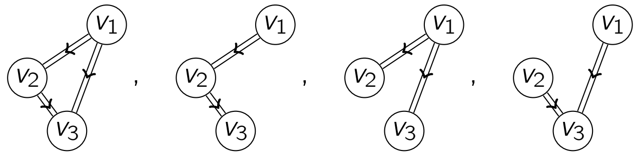
Technical caveat: I will use "nonunital" operads/. . . .

So a *properad* P is a system of $\mathbb{S}_m \times \mathbb{S}_n$ modules $\{P(m, n)\}$ with "composition" maps



$$: P(m_1 + k, n_1) \otimes P(m_2, k + n_2) \to P(m_1 + m_2, n_1 + n_2)$$

$\forall m_1, \ldots, k$ with $k \geq 1$. These should by compatible with $\mathbb{S}$-actions and "associative" for diagrams like



A *dioperad* only has composition maps for $k = 1$.

E.g.: The presentation at left defines Frob_d as a dioperad/properad/prop and $\mathrm{invFrob}_d$ as a properad/prop.

E.g.: When V is a cochain complex, $\mathrm{End}(V) = \{$all operations $V^{\otimes m} \to V^{\otimes n}\}$ is dioperad/properad/prop. When $\dim V < \infty$, it is wheeled.

Remark: I care about $\Omega^\bullet_{\mathrm{dR}}(M)$, which is ∞-dim, so I will not use wheeled things. Also, Koszul duality is harder for wheeled things and props, but easier for operads, dioperads, and properads.

Thm (due to many people, none of them me): The categories {dg operads}, {dg dioperads}, etc. admit model category structure with weak equivalences = quasi-isomorphisms and fibrations = surjections. (The *projective* model structure.)

Technical caveat: characteristic = 0.

Defn: For P an operad, . . . , a *homotopy P-algebra* is a cochain complex V and a homomorphism $\mathbf{h}P \to \mathrm{End}(V)$ for $\mathbf{h}P \overset{\sim}{\twoheadrightarrow} P$ a cofibrant replacement.

3. Quasilocality

To pose the question precisely, we need to answer first:

Pre-question: Which operations on $\Omega^\bullet_{\mathrm{dR}}$ are "local"?

Sufficiently-continuous maps $\Omega^\bullet_{\mathrm{dR}}(M)^{\otimes m} \to \Omega^\bullet_{\mathrm{dR}}(M)^{\otimes n}$ have integral kernels that are distributional de Rham forms on $M^{\times(m+n)}$.

Technical caveat: use projective $\otimes$.

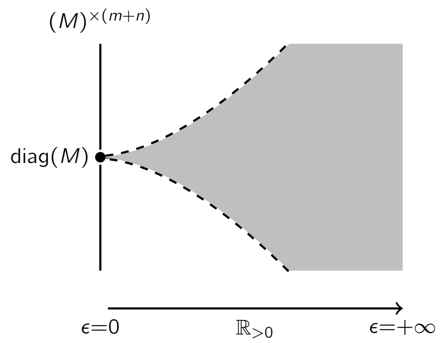
Defn: An operation $\Omega^\bullet_{\mathrm{dR}}(M)^{\otimes m} \to \Omega^\bullet_{\mathrm{dR}}(M)^{\otimes n}$ is *strictly local* if its kernel is supported on $\mathrm{diag}(M) \subseteq M^{\times(m+n)}$.

Wanted defn: A *quasilocal* operation should be supported "near" $\mathrm{diag}(M)$. How near? Arbitrarily near.

Defn: Complex of *homotopy-constant one-parameter families* in a cochain complex V is $\Omega^\bullet_{\mathrm{dR}}(\mathbb{R}_{>0}; V)$. Because the closed elements therein are one-parameter families of homologous closed elements in V.

I will let "ϵ" denote parameter on $\mathbb{R}_{>0}$.

Defn: A *quasilocal operation* is a homotopy-constant one-parameter family of maps $\Omega^\bullet_{\mathrm{dR}}(M)^{\otimes m} \to \Omega^\bullet_{\mathrm{dR}}(M)^{\otimes n}$ that "becomes local" in the $\epsilon \to 0$ limit. I.e. complex of these is $\mathrm{Qloc}(M) \subseteq \Omega^\bullet_{\mathrm{dR}}(\mathbb{R}; \mathrm{End}(\Omega^\bullet_{\mathrm{dR}}(M))$ with integral kernel in (gray part)



for some (white) nbhd of $M^{m+n} \smallsetminus \mathrm{diag}(M)$.

4. Some answers

By triangle inequality, $\mathrm{Qloc}(M)$ is a dioperad and properad. (Not a prop.)

Rmk: $\exists$ canonical map

$$\mathrm{H}^\bullet(\mathrm{Qloc}(M)) \to \mathrm{H}^\bullet(\Omega^\bullet_{\mathrm{dR}}(\mathbb{R}; \mathrm{End}(\Omega^\bullet_{\mathrm{dR}}(M))))$$
$$\cong \mathrm{H}^\bullet(\mathrm{End}(\Omega^\bullet_{\mathrm{dR}}(M))) \cong \mathrm{End}(\mathrm{H}^\bullet_{\mathrm{dR}}(M)).$$

Refined question, dioperad version: Let $\mathbf{h}^{\mathrm{di}}\mathrm{Frob}_d$ denote some cofibrant replacement of Frob_d *as a dioperad*. Is there a homomorphism $\mathbf{h}^{\mathrm{di}}\mathrm{Frob}_d \to \mathrm{Qloc}(M)$ inducing the Frob_d-action on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$?

(Via $\mathrm{H}^\bullet(\mathbf{h}^{\mathrm{di}}\mathrm{Frob}_d) \cong \mathrm{H}^\bullet(\mathrm{Frob}_d) \cong \mathrm{Frob}_d$.)

If so, what is the space of such maps?

Refined question, properad version: Let $\mathbf{h}^{\mathrm{pr}}\mathrm{Frob}_d$ denote some cofibrant replacement of Frob_d *as a properad*. Is there a homomorphism $\mathbf{h}^{\mathrm{pr}}\mathrm{Frob}_d \to \mathrm{Qloc}(M)$ inducing the Frob_d-action on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$?

If so, what is the space of such maps?

Warning: Universal enveloping functor $\{\text{dioperads}\} \to \{\text{properads}\}$ is known to be not exact. So these questions can have different answers.

Refined question, involutive version: Suppose $\chi(M) = 0$. Is there a homomorphism $\mathbf{h}^{\mathrm{pr}}\mathrm{invFrob}_d \to \mathrm{Qloc}(M)$ inducing the $\mathrm{invFrob}_d$-action on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$? (Note: $\mathrm{Frob}_d = \mathrm{invFrob}_d$ for d odd.)

Tools for this type of question:

- Frob_d is Koszul as both a dioperad and a properad. So we have access to a small cofibrant replacement $\mathbf{sh}\mathrm{Frob}_d = \mathrm{Cobar}(\text{Koszul dual of } \mathrm{Frob}_d)$. Ditto $\mathrm{invFrob}_d$.
- Obstruction theory lets you control homotopy groups of $\mathrm{maps}(P, Q)$ by comparing degrees of generators of P to degrees of nonvanishing cohomology of Q.

Theorem, dioperad version: For any M, the space of maps $\mathbf{sh}^{\mathrm{di}}\mathrm{Frob}_d \to \mathrm{Qloc}(M)$ inducing Frob_d-action on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is contractible ($\simeq \{\mathrm{pt}\}$).

"Dioperadic local Poincaré duality is unique."

Proof outline:

(i) Degree calculation $\Rightarrow$ $\mathrm{maps}(\mathbf{sh}^{\mathrm{di}}\mathrm{Frob}_d, \mathrm{Qloc}(M))$ (inducing ...) is contractible if nonempty. It is nonempty iff certain "obstructions" vanish.

(ii) Only generators of $\mathbf{sh}^{\mathrm{di}}\mathrm{Frob}_d$ for which obstructions don't vanish for degree reasons are those controlling associativity, coassociativity, and Frobenius relation. These obstructions vanish (easy calculation). $\square$

Theorem, properad version: For $M = S^1$, the space of maps $\mathbf{sh}^{\mathrm{pr}}\mathrm{Frob}_1 \to \mathrm{Qloc}(S^1)$ inducing Frob_1 str on $\mathrm{H}^\bullet_{\mathrm{dR}}(S^1)$ IS EMPTY.

"Properadic local Poincaré duality is nontrivial."

Proof outline:

(i) Again degree calculation $\Rightarrow$ contractible if nonempty, so must check some obstructions.

(ii) There are three additional ones to check. Two vanish, but the last does not. Explicit (elementary, but not easy) integrals. $\square$

Cor: Dioperad version cannot be strictified.

Remark: When $d \geq 2$, obstruction theory $\Rightarrow$ there are nontrivial higher homotopy groups in space of properadic "local Poincaré dualities" (if this space is nonempty).

5. An abrupt change of topic: infinitesimal Poisson geometry

Defn: An *infinitesimal manifold* is a cocom coalgebra $\mathrm{Sym}(V)$, or rather "coSpec" thereof. Vector space $V =$ "linear chart."

Poisson infinitesimal manifold is $\mathrm{Sym}(V)$ equipped with "Poisson bivector."

Remark: In the world of infinitesimal manifolds, every geometric structure *is* its Taylor series.

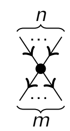
Defn: *DG inf man* is dg coalgebra $(\mathrm{Sym}(V), \partial)$.

Technical caveat: I will only use *pointed* dg inf mans, meaning differential ∂ vanishes at origin, i.e. its Taylor series has no constant term. Then starts with linear term, which defines differential $\partial_V : V \to V$ on V.

Defn: *SH Pois$_d$ dg inf man* is $\mathrm{Sym}(V)$ equipped with L_∞-coalg str on $\mathrm{Sym}(V)[1-d]$ via multi-coderivations.

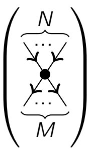
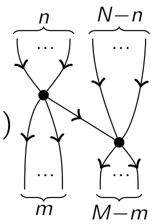
Technical caveat: assume flat (in L_∞ sense) and pointed (all structures vanish at origin).

Exercise: Fix linear part ∂_V. SH Pois$_d$ str on $\mathrm{Sym}(V)$ is same as system of tensors



$: V^{\otimes m} \to V^{\otimes n}$ of hom degree $1 - d(n-1)$

in $(\mathrm{triv}) \otimes (\mathrm{sign})^{\otimes d}$ subrep of $\mathbb{S}_m \otimes \mathbb{S}_n \curvearrowright \hom(V^{\otimes m}, V^{\otimes n})$, satisfying

∂_V () $= \sum_{m,n,M-m,N-n\geq 1} (\#)$ 

#s depend on conventions. Average over permutations of input/output strands, with signs when $d =$ odd.

I.E. an action on V by dioperad $\mathbb{D}^{\mathrm{di}}(\mathrm{Frob}_d)$ or by properad $\mathbb{D}^{\mathrm{pr}}(\mathrm{invFrob}_1)$, where $\mathbb{D} =$ cobar of linear dual.

Remark: By Koszulity, $\mathbb{D}^{\mathrm{di}}(\mathrm{Frob}_d) = \mathbb{D}^{\mathrm{pr}}(\mathrm{invFrob}_d)$ is cofibrant replacement of "quadratic dual," which is LB_d controlling shifted Lie bialgebras with bracket of degree $1-d$ and cobracket of degree 1. I.e.

$$\mathbb{D}^{\mathrm{di}}(\mathrm{Frob}_d) = \mathbb{D}^{\mathrm{pr}}(\mathrm{invFrob}_d) = \mathbf{sh}\mathrm{LB}_d.$$

6. What can you do with local Poincaré duality?

Lemma: $\exists$ canonical map $\mathbb{D}\mathrm{Frob}_0 \to P \otimes \mathbb{D}P$. So if V is P-alg and W is $\mathbb{D}P$-alg, $V \otimes W$ is $\mathbb{D}\mathrm{Frob}_0$-algebra.

Technical caveats: Finiteness constraints on P. For pr-operadic statement, need grading by "genus."

Proof: Sum over diagrams. □

Cor: $\exists$ homotopically-canonical map

$$\mathbb{D}^{\mathrm{di}}\mathrm{Frob}_0 \to \mathbf{sh}^{\mathrm{di}}\mathrm{Frob}_d \otimes \mathbf{sh}\mathrm{LB}_d.$$

Thm: Let M oriented d-dim. Since dioperadic local Poincaré duality is homotopically unique, given sh Pois$_d$ str on dg man $\mathrm{Sym}(V)$, get homotopically-canonical sh Pois$_0$ structure on dg man $\mathrm{Sym}\big(\Omega^\bullet_{\mathrm{dR}}(M) \otimes V\big) =$ "derived space of loc. const. maps $M \to \mathrm{Sym}(V)$". □

Defn: This is the *Poisson AKSZ construction*.

7. Properads and quantization

Pretend you have properadic (quasi)local Poincaré duality on $S^1 = \mathbb{R} \cup \{\infty\}$.

Consider $\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] =$ shifted complex of forms vanishing near ∞. Quasilocality $\Rightarrow$ this has $\mathbf{h}^{\mathrm{pr}}\mathrm{Frob}_1\langle -1\rangle$ alg str: for small enough ϵ, don't reach ∞. ($[] =$ shift. $\langle\rangle =$ *shear*: now multiplication has degree 1 and comultiplication has degree 0.)

Suppose given V in cohom degree 0 and Poisson bracket on power series algebra $\widehat{\mathrm{Sym}}(V)$ cont's for power series topology. (Switched from coalgebras to power series algebras for later expositional clarity.) This makes V into a $\mathbf{sh}\mathrm{LB}_1$-coalgebra, i.e. a $\mathbf{sh}\mathrm{LB}_1\langle 1\rangle$-algbera, i.e. deg(bracket) $= 0$ and deg(cobracket) $= 1$.

Then $\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] \otimes V$ is $\mathbb{D}^{\mathrm{pr}}\mathrm{Frob}_0$.

Thm (Drummond-Cole–Terilla–Tradler): Fix ∂_W. A $\mathbb{D}^{\mathrm{pr}}\mathrm{Frob}_0$ str on W is same as differential on $\widehat{\mathrm{Sym}}(W)[\![\hbar]\!]$ of form $\Delta = \partial_W + \delta$, with $\delta \ll 1$ in filtration (plus vanishing of certain Taylor coefficients).

Cor: $\widehat{\mathrm{Sym}}\big(\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] \otimes V\big)[\![\hbar]\!]$ has differential $\partial_{\mathrm{dR}} + \delta$, since we decided V was in cohom degree 0 (otherwise $\partial_W = \partial_{\mathrm{dR}} + \partial_V$).

Unpacking the formulas: $\hbar^\beta$ term in (m,n)th Taylor coef $\delta : \mathrm{Sym}^m\big(\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] \otimes V\big) \to \mathrm{Sym}^n\big(\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R}) \otimes V\big)$ is given by sum-over-diagrams formula: take all genus-β diagrams; evaluate each in $\mathbf{sh}^{\mathrm{pr}}\mathrm{LB}_1$ to get map $V^{\otimes m} \to V^{\otimes n}$, and in $\mathbf{h}^{\mathrm{pr}}\mathrm{Frob}_1 = \mathbb{D}^{\mathrm{pr}}\mathbf{sh}^{\mathrm{pr}}\mathrm{LB}_1$ to get $\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1]^{\otimes m} \to \Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1]^{\otimes n}$; tensor these, divide by # automorphisms, and sum.

Remark: $\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] \simeq \mathbb{R}$. $\widehat{\mathrm{Sym}}$ is exact. So

$$\big(\widehat{\mathrm{Sym}}(\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] \otimes V)[\![\hbar]\!], \partial_{\mathrm{dR}}\big) \simeq \widehat{\mathrm{Sym}}(V)[\![\hbar]\!].$$

Spectral sequence $\Rightarrow$ this survives $\partial_{\mathrm{dR}} \rightsquigarrow \partial_{\mathrm{dR}} + \delta$. Actually, we can be much more precise:

Homological perturbation lemma:

Suppose given a *retraction* (in any additive category)

$$(A_\bullet, \partial_A) \underset{\phi}{\overset{\iota}{\leftrightarrows}} (B_\bullet, \partial) \circlearrowleft \eta \qquad \begin{array}{l} \iota\phi = \mathrm{id}_H \\ \phi\iota = \mathrm{id}_V - [\partial, \eta] \end{array}$$

and a *perturbation* $\partial \rightsquigarrow \partial + \delta$ with $(\partial + \delta)^2 = 0$. *If* $(\mathrm{id}_B - \delta\eta)$ *is invertible*, get new retraction:

$$(A_\bullet, \tilde{\partial}_A) \underset{\tilde{\phi} = (\mathrm{id} - \eta\delta)^{-1}\phi}{\overset{\tilde{\iota} = \iota(\mathrm{id} - \delta\eta)^{-1}}{\leftrightarrows}} (B_\bullet, \partial + \delta) \circlearrowleft \tilde{\eta} = \eta(\mathrm{id} - \delta\eta)^{-1}$$

with $\tilde{\partial}_A = \partial_A + \iota(\mathrm{id} - \delta\eta)^{-1}\delta\phi$. Note: $(\mathrm{id} - \eta\delta)^{-1} = \mathrm{id} + \eta(\mathrm{id} - \delta\eta)^{-1}\delta$.

Pf: Check some eqns.

Cor: Choose "bump" $f \in \Omega^1_{\mathrm{cpt}}(\mathbb{R})$ with $\int f = 1$. Set $\phi = \widehat{\mathrm{Sym}}(\int \otimes \mathrm{id}_V)$ and $\iota_f = \widehat{\mathrm{Sym}}(\alpha \otimes \mathrm{id}_V)$. Then

$$\tilde{\phi} : \left(\widehat{\mathrm{Sym}}(\Omega^\bullet_{\mathrm{cpt}}(\mathbb{R})[1] \otimes V)[\![\hbar]\!], \partial_{\mathrm{dR}} + \delta\right) \overset{\sim}{\leftrightarrows} \widehat{\mathrm{Sym}}(V)[\![\hbar]\!] : \tilde{\iota}_f$$

Now choose two bumps f, g with support(f) strictly to left of support(g). Consider map

$$*_{f,g} = \tilde{\phi} \circ \odot \circ (\tilde{\iota}_f \otimes \tilde{\iota}_g) : \widehat{\mathrm{Sym}}(V)[\![\hbar]\!]^{\otimes 2} \to \widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$$

where $\odot : \widehat{\mathrm{Sym}}(\cdots)^{\otimes 2} \to \widehat{\mathrm{Sym}}(\cdots)$ is symmetric multiplication.

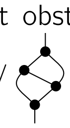
Thm: If this story existed (remember: uses properadic local Poincaré duality!), then

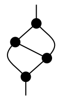
- $\lim_{\epsilon \to 0} *_{f,g}$ converges in power series topology
- to something indep of f, g
- which is unital and associative
- deforming $\odot$ on $\widehat{\mathrm{Sym}}(V)$ in direction of Poisson bracket.

I.E. would have a universal deformation quantization.

Main step of proof: Quasilocality $\Rightarrow$ modulo terms $\ll 1$ in power series topology, for sufficiently small ϵ, $*_{f,g}$ is "homotopically well-defined." I mean: $\odot$ is not a cochain map for differential $\partial_{\mathrm{dR}} + \delta$, but it approximates one on inputs that are far apart from each other.

8. What can we say?

$\nexists\, \mathbf{sh}^{\mathrm{pr}}\mathrm{Frob}_1 \to \mathrm{Qloc}(S^1)$ inducing $\mathrm{Frob}_1 \to \mathrm{End}(\mathrm{H}^\bullet(S^1))$. But only one obstruction, so do have map from maximal subproperad without that obstruction. That maximal subproperad is $\mathbb{D}^{\mathrm{pr}}\left(\mathrm{LB}_1 / \right.$  $\left.\right)$.

Cor (Merkulov, with diff. proof): Suppose $\widehat{\mathrm{Sym}}(V)$ is Poisson s.t. corresponding $\mathbf{h}\mathrm{LB}_1$ structure on V is an $\mathbf{h}\left(\mathrm{LB}_1 / \right.$  $\left.\right)$ structure. (This involves vanishing of certain compositions of Taylor coefs of Poisson bivector, the first in genus 2.) Then $\widehat{\mathrm{Sym}}(V)$ has a universal wheel-free deformation quantization.

Question for the audience: Have you ever seen this relation in Lie bialgebras? Does it have geometric or representation-theoretic meaning?

Defn by analogy:

topological gadget (Ω)	algebraics (A)
$\Omega^\bullet_{\mathrm{cpt,dR}}(\mathbb{R})[1]$	associative algebras
$\Omega^\bullet_{\mathrm{cpt,dR}}(\mathbb{R}^d)[d]$	E_d algebras
$\Omega^{1,\bullet}_{\mathrm{cpt},\bar{\partial}}(\mathbb{C})[1]$	chiral/vertex algebras
$\Omega^\bullet_{\mathrm{cpt,dR}}(\mathbb{R})[1] \otimes \Omega^1_{\mathrm{cpt}}(\mathbb{R})$	chiral conformal nets

Thm modulo checking some details: Suppose you can give V a homotopy P-algebra structure and you can give "topological gadget" Ω a *quasilocal* homotopy $\mathbb{D}^{\mathrm{pr}}P$-algebra structure. Then $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ receives a homotopy "algebraics" A-algebra structure.

Pf: As above: sums over diagrams, HPL. $\square$

Thm: The E_d-algebra you would get from properadic quasilocal Poincaré duality + $\mathrm{LB}_d/\mathrm{Pois}_d$ correspondence is a "universal E_d deformation quantization". I.e. E_d structure deforms $\odot$ in direction of Pois_d bracket. I.e. is an E_d-formality morphism.

Defn: A *formality morphism* of dg algebraic object X is a weak equivalence $f : X \simeq \mathrm{H}^\bullet(X)$ s.t. $\mathrm{H}^\bullet(f) : \mathrm{H}^\bullet(X) \overset{\sim}{\to} \mathrm{H}^\bullet(\mathrm{H}^\bullet(X)) = \mathrm{H}^\bullet(X)$ is identity.

Pre-thm: This gives a homotopy equivalence {properadic quasilocal Poincaré dualities on $\mathbb{R}^d$} $\simeq$ {E_d formality morphisms}.

"Pre-thm" means I know what I need to do, but there's some work. AKA "conjecture."

9. Comparison with physics

$\epsilon \to 0$	large volume limit
$*_{f,g}$	2-point function
$\partial + \delta$	BV differential
$\tilde{\phi}$	expectation value / path integral
$\widehat{\mathrm{Sym}}(\Omega \otimes V)[\![\hbar]\!]$	quantum observables in factorization algebra / AQFT
dioperads	tree-level Feynman diagrams / classical field theory
properads	all-loop-level Feynman diagrams / perturbative quantum field theory