Some comments on Heisenberg-picture QFT, Theo Johnson-Freyd, 12 January 2015. p 1/2.

I. Motivations and definitions

I.1. Schrödinger v. Heisenberg quantum mechanics

Schrödinger	textbook Heisenberg	improved Heisenberg	Schrödinger → Heisenburg functor "End"
"Hilbert" space V	assoc algebra A	assoc algebra A	$A = \mathrm{End}(V)$ (e.g. bounded operators)
$u_t : V \to V$, $t \in \mathbb{R}_{>0}$ s.t. $u_{t_1+t_2} = u_{t_1} u_{t_2}$	autos $f_t : A \overset{\sim}{\to} A$ s.t. $f_{t_1+t_2} = f_{t_1} f_{t_2}$	pointed bimodules s.t. group law	${}_A A_A$ with left action twisted by $a \mapsto u_t a u_1^{-1}$ $\cong$ usual ${}_A A_A$ but pointed by u_t.
distinguished $v_i \in V$, $w_j \in V^*$, $a_k : V \to V$	???	distinguished pointed (bi)modules	${}_A V$ pointed by v_i, $(V^*)_A$ pointed by w_j, ${}_A A_A$ pointed by a_k

I.2. Non-affine quantum spaces

Defn: $\mathrm{VECT}_0 = \mathbb{K}$. $\mathrm{VECT}_1 = \mathrm{VECT}_{\mathbb{K}}$ or $\mathrm{DGVECT}_{\mathbb{K}}$. VECT_2 = bicategory of $\mathbb{K}$-linear cocomplete categories & $\mathbb{K}$-linear cocontinuous functors (strict or $(\infty, 1)$).

Rmk: {cocomplete categories} has set-theoretic difficulties. Empirically, all examples $\in$ subbicat $\mathrm{PRES}_{\mathbb{K}}$ of locally presentable cats. **Thm (Bird, Kelly, . . .):** $\mathrm{PRES}_{\mathbb{K}}$ is closed under most categorical constructions.

Defn: Let X be a "space" (scheme or manifold or . . .). $\mathcal{O}_k(X)$ = sym mon k-cat of functions $X \to \mathrm{VECT}_k$.
E.g.: $\mathcal{O}_0(X) = \mathcal{O}(X)$. $\mathcal{O}_1(X) = \mathrm{QCOH}(X)$.

Rmk: $\mathcal{O}_{k-1}(X) = \mathrm{End}_{\mathcal{O}_k(X)}(\text{unit object})$.

Defn: X is *k-affine* if $X \to \mathrm{Spec}(\mathcal{O}_k(X))$ is an equiv. **Rmk:** "$\mathrm{Spec}(\mathcal{C})$" is a stack in fpqc topology.

Gelfand–Naimark theorem: Most spaces from functional analysis and point-set topology are 0-affine.

But: Most spaces from alg geo are not 0-affine.

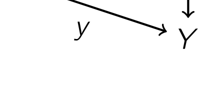
Tannakian philosophy: Most spaces from algebraic geometry are 1-affine.

E.g.: 1-affine spaces $\supseteq$ qcqs schemes (Brandenburg–Chirvasitu), affine ind-schemes (Brandenburg–Chirvasitu–JF), Nötherian algebraic stacks with affine stabilizers (Hall–Rydh). **Non-e.g.:** Algebraic stacks with non-affine stabilizers are not 1-affine (Hall–Rydh).

Defn: A *k-quantum space* X is something with $\mathcal{O}_0(X)$ a k-algebra ($= E_k$-alg). Spec(a k-alg) is 0-*affine*.

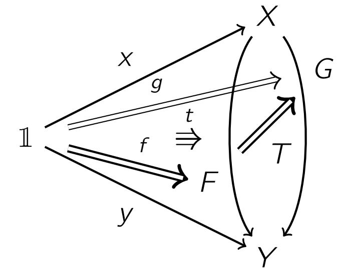
Rmk: If A is k-alg, MOD_A is $(k-1)$-monoidal category.
Defn: Spec(a $(k-1)$-mon cat) is 1-*affine k-quantum*.

Bicat of k-affine k-quantum spaces $\mathbf{ALG}_0(\mathbf{VECT}_k)$:
Objects: Pairs $X \in \mathrm{VECT}_k$, $x \in X$.
1-morphisms: *Lax homs* in sense of JF–Scheimbauer:



, i.e. $F : X \to Y$, $f : y \to F(x)$.

2-morphisms:



Etc.

$\mathbf{ALG}_0(\mathbf{VECT}_2)$ versus $\mathbf{ALG}_1(\mathbf{VECT}_1)$: Latter has objects = assoc algs, 1-morphisms = pointed bims, 2-morphisms = pointed intertwiners. Related by functor "MOD":
$A \mapsto (\mathrm{MOD}_A, A_A)$.
$\big({}_A M_B, m\big) \mapsto \big((-) \otimes_A M, m : B_B \to M_B\big)$.

$\mathrm{MOD} \circ \mathrm{End}$:
$V \mapsto \big(\mathrm{MOD}_{\mathrm{End}(V)}, \mathrm{End}(V)_{\mathrm{End}(V)}\big) \cong \big(\mathrm{VECT}_1, V\big)$.
$(f : V \to W) \mapsto \mathrm{MOD}\big({}_{\mathrm{End}(V)} \hom(V, W)_{\mathrm{End}(W)}, f\big)$
$\cong \big(\mathrm{id}_{\mathrm{VECT}_1}, f : V \to W\big)$.

Pre-thm (JF–Scheimbauer): For any sym mon (∞, k)-category $\mathcal{V}$ (with mild good properties) there is a sym mon (∞, n)-category $\mathrm{ALG}_{n-k}(\mathcal{V})$ whose $0, \ldots, n-k$ dimensions are as in Calaque–Scheimbauer's construction, and whose $n-k, \ldots, n$ dimensions are as in $\mathrm{ALG}_0(\mathcal{V})$.

I.3. Heisenberg-picture QFT

Choose an n-category "$\mathrm{BORD}^G_{d-n,\ldots,d}$" of "bordisms in dims $d-n$ through d with geometry G." E.g. G = Riemannian metric. (Equip bordisms with germs of d-dim collars; those collars are what have G-geometry.)

Defn: A *Schrödinger picture QFT* for $\mathrm{BORD}^G_{d-n,\ldots,d}$ is a sym mon functor $\mathrm{BORD}^G_{d-n,\ldots,d} \to \mathrm{VECT}_n$.

Defn: A *k-affine Heisenberg picture QFT* for $\mathrm{BORD}^G_{d-n,\ldots,d}$ is a sym mon functor $\mathrm{BORD}^G_{d-n,\ldots,d} \to \mathrm{ALG}_{n-k}(\mathrm{VECT}_k)$.

Heisenberg = twisted (aka relative): When $k = n$, functors $\mathrm{BORD}^G_{d-n,\ldots,d} \to \mathrm{ALG}_0(\mathrm{VECT}_n)$ are *oplax relative QFTs*, i.e. sym mon oplax natural trans with domain the constant functor $\mathbb{1} : \mathrm{BORD}^G_{d-n,\ldots,d} \to \mathrm{VECT}_n$, in the sense of JF–Scheimbauer.

Rmk: Since I like strict things, henceforth I mostly assume $k = 2$, $\mathrm{VECT}_2 = \mathrm{PRES}_{\mathbb{K}}$, and $n \leq 3$. I can build 4-cat $\mathrm{ALG}_2(\mathrm{PRES}_{\mathbb{K}})$ without appleaing to above pre-thm.

II. Examples and non-examples

II.1. General TQFT comments

Special case of geometry is G = fr = "framing." Recall *Cobordism Hypothesis* (Lurie): $\{\text{qfts } \mathrm{BORD}^{\mathrm{fr}}_{0,\dots,d} \to \mathcal{C}\} \simeq \{d\text{-dualizable objects in } \mathcal{C}\}$. Such QFTs are called *(framed) topological*, aka *(framed) TQFTs*. I will also use G = or = orientation, giving *oriented TQFTs*.

Example (Calaque–Scheimbauer): Every object of $\mathrm{ALG}_d(\mathcal{V})$ is d-dualizable, so get lots of 0-affine Heisenberg-picture TQFTs.

Non-example (JF–Scheimbauer): If $\mathcal{V}$ is an (∞, k)-category, the groupoid of d-dualizable objects in $\mathrm{ALG}_{d-k}(\mathcal{V})$ is contractible — just $\{\mathbb{1}\}$. So Heisenberg-picture TQFTs do not extend one dimension higher to some kind of Schrödinger-picture TQFTs.

II.2. A non-topological example & a relation to another approach to "Heisenberg picture QFT"

Defn: $\mathrm{EMB}^G_d = (\infty,1)$-category of d-dim manifolds with G-geometry, with morphisms = embeddings. A *factorization algebra* is a sym mon functor $F : \mathrm{EMB}^G_d \to \mathrm{VECT}$ which is local for the Weiss topology.

Example (Dwyer–Stolz–Teichner, as seen through my glasses): Given factorization algebra $F : \mathrm{EMB}^G_d \to \mathrm{VECT}$, get 1-affine Heisenberg-picture QFT $\mathrm{BORD}^G_{d-1,d} \to \mathrm{ALG}_0(\mathrm{PRES}_{\mathbb{K}})$. When F is locally constant (i.e. topological), this QFT is affine, and reproduces (MOD of) Calaque–Scheimbauer's TQFT.

Rmk: My understanding is their construction is extended & derived. I haven't grokked those parts yet.

II.3. 1-dimensional TQFTs

Exercise: If $(\mathcal{C}, C) \in \mathrm{ALG}_0(\mathrm{VECT}_2)$ is 1-dualizable, then C is compact projective.

Corollary: Let X be a scheme. $(\mathrm{QCOH}(X), \mathcal{O}_X) \in \mathrm{ALG}_0(\mathrm{VECT}_2)$ is 1-dualizable iff X is (0-)affine.

The 1-dim TQFT defined by $(\mathrm{QCOH}(X), \mathcal{O}_X)$ is "classical 1-dim topological sigma model with target X."

Can non-0-affine $\mathrm{QCOH}(X)$s still be "twists"? No:

Non-example (Brandenburg–Chirvasitu–JF): Let X be a scheme. If X contains a closed projective subscheme, then $\mathrm{QCOH}(X)$ is not 1-dualizable in $\mathrm{PRES}_{\mathbb{K}}$.

(Non-)example (Brandenburg–Chirvasitu–JF): Let C be a coassociative coalgebra. COMOD^C is 1-dualizable in $\mathrm{PRES}_{\mathbb{K}}$ iff it has enough projectives.

If $C = \mathcal{O}(G)$ for G an affine algebraic group, corresponding TQFT is "classical 1-dim topological gauge theory". $\mathrm{COMOD}^{\mathcal{O}(G)} = \mathrm{REP}(G)$ has enough projectives when reductive in char 0, but not when solvable, nor when semisimple in char p.

Example (Brandenburg–Chirvasitu–JF): If X is affine and G is (virtually) linearly reductive, then $\mathrm{QCOH}([X/G])$ is 1-dualizable in $\mathrm{PRES}_{\mathbb{K}}$. The TQFT is "topological gauge theory coupled to matter."

Example (Brandenburg–Chirvasitu–JF): If $\mathcal{C} \in \mathrm{PRES}_{\mathbb{K}}$ has enough compact projectives, then it is 1-dualizable.

Conj (Brandenburg–Chirvasitu–JF): This is iff. **Subconj:** $\mathrm{QCOH}(X)$ is 1-dualizable iff X is affine.

Example (Ben-Zvi–Francis–Nadler): In derived world, only need enough compacts. Then all of these non-examples become examples.

II.4. 3-dimensional TQFTs

Example (Walker, as seen through my glasses):

- $\mathcal{C} \in \mathrm{ALG}_1(\mathrm{PRES}_{\mathbb{K}})$ is 2-dualizable (i.e. defines 2-dim framed TQFT) if unit object $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ is compact projective and $\mathcal{C}$ is generated by its dualizable objects.
- $\mathcal{C} \in \mathrm{ALG}_1(\mathrm{PRES}_{\mathbb{K}})$ is moreover SO(2)-invariant (i.e. defines 2-dim oriented TQFT) if it is moreover equipped with a "pivotal" structure.
- $\mathcal{C} \in \mathrm{ALG}_2(\mathrm{PRES}_{\mathbb{K}})$ is 3-dualizable if $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ is compact projective and $\mathcal{C}$ is generated by its dualizable objects.
- $\mathcal{C} \in \mathrm{ALG}_2(\mathrm{PRES}_{\mathbb{K}})$ is moreover SO(3)-invariant if it is moreover equipped with a "ribbon" structure.

I believe, but cannot yet prove, that these are sharp.

In particular: $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$. $\mathcal{C}$ = TL = cocompletion of Temperley–Lieb category = rep thy of Temperley–Lieb algebra. Corresponding tqft $M \mapsto \int_M \mathrm{TL}$ packages together all objects of Kauffman-bracket skein theory (skein algebras, relative skein modules, DAHA, ...).

Rmk: One power of $\mathrm{PRES}_{\mathbb{K}}$ is that all constructions are well-behaved for "tensoring"-type operations. In particular, for any commutative ring hom $\mathbb{Z}[q, q^{-1}] \to R$, there is a category $\mathrm{TL} \otimes_{\mathbb{Z}[q,q^{-1}]} R \in \mathrm{ALG}_2(\mathrm{PRES}_R)$, and $(\int_M \mathrm{TL}) \otimes_{\mathbb{Z}[q,q^{-1}]} R \cong \int_M (\mathrm{TL} \otimes_{\mathbb{Z}[q,q^{-1}]} R)$.

E.g.: $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ via $q \mapsto -1$. Then $\mathrm{TL} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} = \mathrm{REP}(\mathrm{SL}(2))$. This "classical limit" is $M \mapsto \mathrm{QCOH}$(stack of SL(2)-local systems on M).

For general q, the TQFT is "*quantum* topological SL(2) gauge theory" aka SL(2) Chern–Simons theory.

Optimism: This might help answer various "semiclassics" questions from quantum topology, e.g. "AJ conjecture" relating colored Jones polynomial to character variety.

Using this technology, I am close to proving that the colored Jones polynomial distinguishes the unknot. Check back in a couple months.