Work in progress. Based in part on jt work w/ Davide Gaiotto and in part on jt work w/ Mike Hopkins.

0.1. Mathematical Motivation

Fund Thm of Alg (conj Roth 1608, Girard 1629; proved Argand 1813, Gauss 1816): Among fd com $\mathbb{R}$-algs, $\mathbb{C}$ is distinguished noncanonically by:

- $\forall A \neq 0$ fd com $\mathbb{R}$-alg, every map $\mathbb{C} \to A$ is injective.
- $\forall A \neq 0$ fd com $\mathbb{R}$-alg, $\exists\ A \to \mathbb{C}$.

Furthermore, $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C}) = \mathbb{Z}/2$ and $\mathbb{R} \xrightarrow{\sim} \mathbb{C}^{\mathbb{Z}/2}$.

Existence of fiber functors (Deligne 2002): Among fd sym-$\otimes$ $\mathbb{R}$-lin cats, $\mathrm{SVEC}_{\mathbb{C}}$ is distinguished noncan'ly by:

- $\forall \mathcal{A} \neq 0$ fd sym-$\otimes$ cat, every $\mathrm{SVEC}_{\mathbb{C}} \to \mathcal{A}$ is injective on homotopy (faithful & inj on iso classes of objects).
- $\forall \mathcal{A} \neq 0$ fd sym-$\otimes$ cat, $\exists\ \mathrm{SVEC}_{\mathbb{C}} \to \mathcal{A}$.

$\mathrm{Aut}_{\mathbb{R}}(\mathrm{SVEC}_{\mathbb{C}}) = \mathbb{Z}/2 \times B(\mathbb{Z}/2)$ and $\mathrm{VEC}_{\mathbb{R}} \xrightarrow{\sim} (\mathrm{SVEC}_{\mathbb{C}})^{\mathrm{Aut}}$.

Conj (H–JF): For fd sym-$\otimes$ $\mathbb{R}$-lin 2-cats, use $\mathrm{SALG}_{\mathbb{C}}$.
N.B. $\mathrm{Aut} = \mathbb{Z}/2 \times B(\mathbb{Z}/2)$ and $\mathrm{SALG}_{\mathbb{R}} \xrightarrow{\sim} (\mathrm{SALG}_{\mathbb{C}})^{\mathrm{Aut}}$

Conj (Ostrik 2015): In char $p \geq 3$, use $\mathrm{VER}_p = \mathrm{SU}_q(2)$ at level $k = p-2$. **N.B.** For $p \geq 5$, $\mathrm{SVEC} \to \mathrm{VER}$ is purely inseparable. **Conj** (Delign 2015): $p = 2$ case.

0.2. Physical Motivation

Classification of states of matter? Way too hard.

Defn: *Gapped topological* matter is almost trivial: finite-degeneracy ground states, indep of size of system.

Conj (Kapustin 2014):

- $\{$bosonic gapped top matter$\}^\times = \Sigma I_{\mathbb{Z}}\mathrm{MSO}$
- $\{$fermionic gapped top matter$\}^\times = \Sigma I_{\mathbb{Z}}\mathrm{MSpin}$

Notes: $\Sigma I_{\mathbb{Z}} =$ (shifted) Anderson duality. MSO, MSpin = cobordism spectra. $\{\text{matter}\}^\times = \Omega$-spectrum of *invertible* matter. Ω-spectrum = algebraic topologist's version of abelian group. C.f. Kitaev 2013, G–JF 2017.

Why CPT and spin-stats hold for condensed matter? Thms only in high energy relativistic theory. C.f. JF 2017.

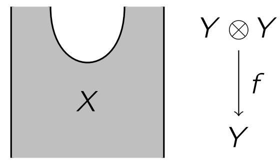
1.1. Condensations

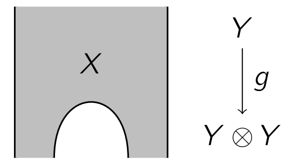
How to study nd matter? Slice into $(n-1)$d slices. Terrible method if nd matter not gapped top (otherwise, thickness of slice is important). Bad method if slice isn't gapped top (otherwise, too hard).

Slice is gapped top when nd matter admits gapped top b.c. (Some gapped top systems, e.g. 3d Chern–Simons thy, fail.) Physics of slice depends on choice of b.c.

Assume X admits gapped top b.c. $Y = (n-1)$d slice. Data of X = data of Y + how to reform the bonds that were broken when slicing X into a stack of Ys.

Idea: breaking/reforming bonds is *adiabatic* (takes ground states to ground states). Specifically, get morphisms:





Exercise: In 2d, f, g make Y into nonunital Frob alg.

Fact: Moreover, in 2d, is *special* Frob alg: $fg = \mathrm{id}_Y$.

Defn (G–JF): A *condensation* in a 2-cat is a 1-morphism Y with a split surjection $Y^2 \twoheadrightarrow Y$ plus associativity.

Defn: If $\mathcal{C}$ is sym-$\otimes$ n-cat (e.g. $\mathcal{C} = \{(n-1)$d gapped top systems$\}$, $B\mathcal{C} =$ sym-$\otimes$ $(n+1)$-cat with objects$=\{1\}$ and $\mathrm{End}(1) = \mathcal{C}$.

Special Frob algs = condensations in $B\mathrm{VEC}$.

1.2. Absolute limits

Consider a 1-cat $\mathcal{C}$. An *idempotent* is an endo $e : M \to M$ such that $e^2 = e$. TFAE:

- $\lim\{\ M \circlearrowleft e\ \} = N$ exists.
- $\mathrm{colim}\{\ M \circlearrowleft e\ \} = N$ exists.
- $\exists$ factorization $e = ip$ s.t. $pi = \mathrm{id}_N$.

Cor: $\forall F : \mathcal{C} \to \mathcal{D}$, $\lim(F(e)) = F(\lim(e))$. I.e. (limits of) idempotents are *absolute*.

In lin cat, $\oplus$ is also *absolute*: lim & colim & eqn.

It is relatively easy to add to a 1-cat $\mathcal{C}$ all its ab lims. Called *Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$.

In an n-cat, condensations are also absolute.

Expect: condensation and $\oplus$ are the only ab lims.

Defn: For sym-$\otimes$ n-cat $\mathcal{C}$, *suspension* $\Sigma\mathcal{C} = \mathrm{Kar}(B\mathcal{C})$.

E.g. (folklore): $\Sigma A \simeq \{$f.g. proj A-modules$\} = \mathrm{MOD}_A^{\mathrm{fd}}$.

fd = "fully dualizable." C.f. Lurie Cobord Hyp.

Thm (G–JF): $\Sigma^2 A \simeq \{$sep A-algebras, f.g. bims$\} = \mathrm{ALG}_A^{\mathrm{fd}}$.

Nontrivial: Diff Frob structures are "Frob-Morita equiv."

Expect: $\Sigma^n\mathbb{R} = \{$fd n-vector spaces$\}$.

Thm (G–JF): $\Sigma^n A \simeq \{A$-linear nd gapped top matter that can be condensed from the vacuum$\}$. Given $X \in \Sigma^n A$, give commuting projector Hamiltonian description.

2.1. Technical aside: towers

Defn: An $\mathbb{R}$-linear *tower* $\mathcal{C}_\bullet$ consists of:

- for each n, a weak n-cat $\mathcal{C}_n$,
- for each n, a distinguished object $1 \in \mathcal{C}_n$,
- equivs $\Omega\mathcal{C}_n = \mathrm{End}_{\mathcal{C}_n}(1) \simeq \mathcal{C}_{n-1}$,
- such that each $\mathcal{C}_n$ is $\mathbb{R}$-linear,
- and each $\mathcal{C}_n$ is closed for ab lims.

E.g.: If A is a com $\mathbb{R}$-alg, $\Sigma^\bullet A$ is a tower.

Towers are the cat version of Ω-spectra. Each $\mathcal{C}_n$ is an "infinite loop n-cat," i.e. sym-$\otimes$. Towers are the "commutative rings" of ∞-*categorical algebra*.

2.2. Invitation to ∞-categorical Galois theory

Main conj (JF): $\exists$ $\mathbb{R}$-linear tower $\mathcal{R}_\bullet$ distinguished noncanically by:

- $\forall \mathcal{A}_\bullet \neq 0$ fd, every $\mathcal{R}_\bullet \to \mathcal{A}_\bullet$ is an inj on homotopy (i.e. each $\mathcal{R}_n \to \mathcal{A}_n$ is an inj on equiv classes of objects).
- $\forall \mathcal{A}_\bullet \neq 0$ fd, $\exists$ map $\mathcal{A}_\bullet \to \mathcal{R}_\bullet$.

Summary: $\mathcal{R}$ is an *alg closed ∞-categorical field*, the *∞-cat alg closure of* $\mathbb{R}$.

Main conj cont: Furthermore, the *∞-cat absolute Galois group of* $\mathbb{R}$ is $\mathrm{Aut}_{\mathbb{R}}(\mathcal{R}_\bullet) \simeq \mathrm{O}(\infty)$, and $\Sigma^\bullet\mathbb{R} \xrightarrow{\sim} \mathcal{R}_\bullet^{\mathrm{O}(\infty)}$.

E.g.: $\mathcal{R}_0 = \mathbb{C}$. $\mathcal{R}_1 = \mathrm{SV{\small EC}}_{\mathbb{C}}$.

Lemma: $\mathcal{R}_2 = \mathrm{SA{\small LG}}_{\mathbb{C}}$, proving H–JF conj.

Pf: By G–JF thm, $\mathrm{SA{\small LG}}_{\mathbb{C}} = \Sigma\mathrm{SV{\small EC}}_{\mathbb{C}}$. By Deligne thm, $\mathrm{SV{\small EC}}_{\mathbb{C}}$ is Galois over $\mathbb{R}$ with Galois group $O(\infty)/\mathrm{Spin}(\infty)$. So $\mathcal{R}_\bullet$ is Galois over $\mathrm{SV{\small EC}}_{\mathbb{C}}$ with Galois group $\mathrm{Spin}(\infty)$, i.e. $\mathcal{R}_\bullet^{\mathrm{Spin}(\infty)} = \Sigma^{\bullet-1}\mathrm{SV{\small EC}}_{\mathbb{C}}$. Since $\pi_\bullet\mathrm{Spin}(\infty) = 0$ for $\bullet \geq 2$, $\mathcal{R}_\bullet^{\mathrm{Spin}(\infty)} = \mathcal{R}_\bullet$ for $\bullet \leq 2$. $\square$

But $\mathcal{R}_3$ has new mathematics, since $\pi_3\mathrm{Spin}(\infty) \neq 0$.

Guess: $\mathcal{R}_3$ can be described in terms of the "Morita theory" of (rational) vertex operator algebras.

2.3. Relation to physics

My hope is that $\mathcal{R}_\bullet$ is the space of all gapped top systems, not just those with gapped top boundary.

Physical arguments suggest that 3d gapped top systems always admit RCFT boundary, hence the guess for $\mathcal{R}_3$.

If guess is correct, requires non-top equivs of RCFTs. I suspect these come from "supersymmetric twisting." Currently looking for appropriate twists with my student Jessica Weitbrecht.

2.4. Relation to cobordism spectra

Pre-thm (aka conj + good idea for pf): If $\mathcal{R}_\bullet$ exists, then $\forall \mathcal{A}_\bullet$ fd tower, $\pi_0 \hom(\mathcal{A}_\bullet, \mathcal{R}_\bullet) = \hom(\mathcal{A}_0, \mathcal{R}_0)$.

Cor of pre-thm: $\mathcal{R}_\bullet^\times = I_{\mathbb{C}^\times}\mathbb{S}$.

$\mathcal{R}_\bullet^\times = \otimes$-inv objects in $\mathcal{R}_\bullet$. $\mathbb{S}$ = stable sphere spectrum. $I_{\mathbb{C}^\times}$ = Brown–Comenetz duality. Feature: $\pi_{-n} I_{\mathbb{C}^\times} T = \hom(\pi_n T, \mathbb{C}^\times)$. Universal property of $I_{\mathbb{C}^\times}\mathbb{S}$: for any (nice) spectrum T, $\hom(T, I_{\mathbb{C}^\times}\mathbb{S}) = \hom(\pi_0 T, \mathbb{C}^\times)$.

Pf of cor: Given spectrum T, set $\mathcal{A} = \Sigma^\bullet\mathbb{R}[T]$ = "group algebra of T". Then $\hom(T, \mathcal{R}_\bullet^\times) = \hom(\Sigma^\bullet\mathbb{R}[T], \mathcal{R}_\bullet) = \hom(\mathbb{R}[\pi_0 T], \mathbb{C}) = \hom(\pi_0 T, \mathbb{C}^\times)$. $\square$

Cor of cor: By taking fixed points, and working "up to phase" (meaning: $\mathbb{C}^\times \rightsquigarrow S^1$, and so $I_{\mathbb{C}^\times} \rightsquigarrow \Sigma I_{\mathbb{Z}}$), G-JF thm gives Kapustin conj:

- $(\Sigma^\bullet\mathbb{R})^\times \simeq \Sigma I_{\mathbb{Z}}\mathrm{MO}$
- $(\Sigma^\bullet\mathbb{C})^\times \simeq \Sigma I_{\mathbb{Z}}\mathrm{MSO}$
- $(\Sigma^\bullet\mathrm{SV{\small EC}}_{\mathbb{C}})^\times \simeq \Sigma I_{\mathbb{Z}}\mathrm{MSpin}$

2.5. Higher spin-statistics

A *hermitian* QFT is one where Time Reversal is identified with Complex Conjugation. A *spin-statistics* QFT is one where is identified with $(-1)^f$. N.b.: = (belt trick). In high dimensions, these are *structure*.

For TFTs, can define these in terms of "tangential structures" valued not in spaces but in "affine categorified schemes" like $\mathrm{Spec}(\mathrm{SV{\small EC}}_{\mathbb{C}})$ or $\mathrm{Spec}(\mathcal{R}_\bullet)$.

Thm (JF 2017): If spin super $\mathbb{C}$-linear TFT $\mathcal{Z}$ is *positive* in sense that the real unoriented TFT $\int_{\text{spin structures}} \mathcal{Z}$ is positive, then $\mathcal{Z}$ admits canonical *up to contractible space of choices* hermitian and spin-statistics structures.

Fundamental reason: The "Galois" and "Cobordism" actions of $O(\infty)$ on $\mathcal{R}_\bullet$ are almost, but not quite, identified. Compare: $z \mapsto z^{-1}$ and $z \mapsto z^*$ are homotopy-equivalent actions of $\mathbb{Z}/2$ on $\mathbb{C}^\times$.

So the Conjecture predicts a *higher spin-statistics theorem*, saying that *positive* (aka unitary, aka reflection-positive) QFTs admit canonical-up-to-contractible-space identifications of the "Galois" action of $O(n) \subset O(\infty)$ with the action by rotating spacetime.

3. Conclusion: a picture of a cobordism