See `arXiv:1502.06526`, joint w/ C. Scheimbauer.

1. Functorial qfts: Schrödinger v. Heisenberg

Spacetime categories: QFTs associate algebraic data ("partition functions", "OPEs", ...) to geometric data ("spacetimes" equipped with some geometric structure). This should be *local* when spacetimes are cut and pasted.

How to organize "locality"? One way is to build an (∞, n)-category SPACETIMES whose k-dim morphisms are k-manifolds surrounded by germs of n-dimensional manifolds equipped with desired geometric structure.

Henceforth: fix choice of SPACETIMES.

Schrödinger-picture qft: The *Schrödinger picture* of quantum mechanics uses systems described by Hilbert spaces and linear maps between them. Atiyah (et al.): a *Schrödinger picture n-dimensional qft* is a symmetric monoidal (∞, n)-functor Z : SPACETIMES $\to$ VECT_n, where $\text{VECT}_n = \{$linear $(\infty, n-1)$-categories$\}$ is an (∞, n)-category whose $(n-1)$-morphisms are vector spaces (maybe topological, chain complexes, etc.).

Problem: Hilbert spaces aren't physical. "Space of states" is $\mathbb{P}\mathcal{H}$, not $\mathcal{H} \in$ VECT. But do want $\otimes, \oplus$.

Solution: Pointed category $(\mathcal{H}, \text{VECT})$ has

$$\text{Aut}((\mathcal{H}, \text{VECT})) \cong \text{PGL}(\mathcal{H}) = \text{Aut}(\mathbb{P}\mathcal{H}).$$

Remark: Often, $(\mathcal{H}, \text{VECT}) \simeq (A, \text{MOD}_A)$ in bicategory of pointed categories, i.e. is *affine*.

Heisenberg-picture qft: Should assign pointed categories to spaces. What about spacetimes, etc.?

Heisenberg = relative: A pointed category $(C, \mathcal{C})$ is a morphism $C : \mathbb{1} \to \mathcal{C}$, where $\mathbb{1}$ is the monoidal unit in the bicategory of categories. So *Heisenberg-picture qft* could be a "morphism" $Z : \mathbb{1} \Rightarrow \mathcal{Z}$ where $\mathbb{1}$: SPACETIMES $\to \text{VECT}_{n+1}$ is constant functor and $\mathcal{Z}$: SPACETIMES $\to \text{VECT}_{n+1}$ is a "categorified Schrödinger picture qft," called the *twist*.

Guess: "Morphism" = "sym $\otimes$ natural transformation."

Exercise: For topological 1-bordisms cat BORD_1, all sym $\otimes$ "strong" nat trans between functors $\text{BORD}_1 \to \mathcal{C}$ are isomorphisms, for any sym $\otimes$ bicategory $\mathcal{C}$.

Defn: An *(op)lax transformation* between functors of bicategories is like a nat trans, but "naturality" is just a morphism, not nec. invertible. In a *strong* transformation, naturality is imposed by isomorphisms.

2. What is an (∞, n)-category?

$\exists$ many equivalent models of "(∞, n)-category." I will describe "complete n-fold Segal space" model.

Warm-up: A *category* is a simplicial set $\mathcal{C}_\bullet : \Delta^{\text{op}} \to$ SETS such that ("Segal condition") $\forall k \geq 1$, the map

$$\mathcal{C}_k \to \underbrace{\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1}_{k \text{ times}}$$

is an iso. (Use $\{(i-1, i) : \Delta^1 \to \Delta^k\}_{i=1}^k$.)

A *functor* is a map of simplicial sets. BUT: this gets WRONG notion of "equivalence" (no isos of functors).

Defn: An $(\infty, 1)$-*category* is a simplicial topological space $\mathcal{C}_\bullet : \Delta^{\text{op}} \to$ SPACES such that $\forall k \geq 1$, the map

$$\mathcal{C}_k \to \underbrace{\mathcal{C}_1 \times^h_{\mathcal{C}_0} \mathcal{C}_1 \times^h_{\mathcal{C}_0} \cdots \times^h_{\mathcal{C}_0} \mathcal{C}_1}_{k \text{ times}}$$

is a homotopy equivalence ($\times^h_{\mathcal{C}_0}$ is homotopy fibered product), AND MOREOVER ("completeness") every (family of) invertible-up-to-homotopy 1-morphism(s) is homotopic to an (family of) identity 1-morphism(s).

A *strict functor* is a map of simplicial spaces. Note: $\hom(\mathcal{B}_\bullet, \mathcal{C}_\bullet)$ is automatically a space. A functor is an *equivalence* if it is levelwise so. Localize at equivalences. This gets the RIGHT notion.

Remark: Model category theory makes this all clean. Rather than localizing, work just with $(\infty, 1)$-categories that as simplicial spaces are cofibrant for projective model structure. Or work just with fibrants for injective model.

Defn: An (∞, n)-*category* is an n-fold simplicial space $\mathcal{C}_{\vec{\bullet}} : (\Delta^{\times n})^{\text{op}} \to$ SPACES which is $(\infty, 1)$-cat in each direction (i.e. an $(\infty, 1)$-cat internal to ... internal to $(\infty, 1)$-cats), AND MOREOVER $\forall i \leq n$, $\forall \vec{k} \in \mathbb{N}^i$, $\mathcal{C}_{\vec{k},0,\bullet\cdots\bullet} \cong \mathcal{C}_{\vec{k},0,0\ldots 0}$ (i.e. at every level, categories of objects are actually spaces). So $\mathcal{C}$ is built from the spaces $\mathcal{C}_{(i)} = \mathcal{C}_{\underbrace{1,\ldots,1}_{i \text{ 1s}},0,\ldots,0} = \{i\text{-morphisms in } \mathcal{C}\}$.

Strict functors are maps of n-fold simplicial spaces. *Equivalences* are levelwise. Localize at those. Or use model category theory.

Defn: A *symmetric monoidal* (∞, n)-*category* is $\mathcal{C}^{\otimes}_{\vec{\bullet}}(\bullet) : (\Delta^{\times n} \times \Gamma)^{\text{op}} \to$ SPACES (Segal's Γ is opposite to $\{$finite pointed sets$\}$), which is (∞, n)- in the $\Delta^{\times n}$ direction, and $\forall k$, $\mathcal{C}^{\otimes}_{\vec{\bullet}}(k) \to \mathcal{C}^{\otimes}_{\vec{\bullet}}(1)^{\times k}$ is a homotopy equivalence.

Strict functors, equivalences, localize, model categories.

3. Walking morphisms and the main definition

For (∞, n)-categories $\mathcal{B}, \mathcal{C}$, let $\mathrm{maps}(\mathcal{B}, \mathcal{C})$ = *derived* space of functors = mapping space in the homotopy category = hom(cofibrant replacement of $\mathcal{B}$, fibrant replacement of $\mathcal{C}$).

$\{(\infty, n)\text{-categories}\}$ is *Cartesian closed*: $\forall \mathcal{B}, \mathcal{C}$, $\exists$ an (∞, n)-category $\underline{\mathrm{maps}}(\mathcal{B}, \mathcal{C})$ s.t.

$$\mathrm{maps}(-, \underline{\mathrm{maps}}(\mathcal{B}, \mathcal{C})) \simeq \mathrm{maps}((-) \times \mathcal{B}, \mathcal{C}).$$

Its space of objects $\underline{\mathrm{maps}}(\mathcal{B}, \mathcal{C})_{(0)}$ is just $\mathrm{maps}(\mathcal{B}, \mathcal{C})$. Its space of 1-morphisms $\underline{\mathrm{maps}}(\mathcal{B}, \mathcal{C})_{(1)}$ is the space of *strong natural transformations*.

Defn: The *walking 0-morphism* is $\Theta^{(0)} = \{\bullet\}$. The *walking 1-morphism* is $\Theta^{(1)} = \{\bullet \to \bullet\}$. The *walking 2-morphism* is $\Theta^{(2)} = \left\{\bullet \underset{\to}{\overset{\to}{\Downarrow}} \bullet\right\}$. Etc. These satisfy:

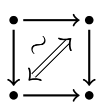
$$\mathcal{C}_{(i)} \simeq \mathrm{maps}(\Theta^{(i)}, \mathcal{C}).$$

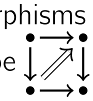
So $\underline{\mathrm{maps}}(\mathcal{B}, \mathcal{C})_{(1)} \simeq \mathrm{maps}(\Theta^{(1)}, \underline{\mathrm{maps}}(\mathcal{B}, \mathcal{C}))$
$\simeq \mathrm{maps}(\Theta^{(1)} \times \mathcal{B}, \mathcal{C}) \simeq \mathrm{maps}(\mathcal{B}, \underline{\mathrm{maps}}(\Theta^{(1)}, \mathcal{C}))$.

I.e. a strong transformation η assigns:

$$B \in \mathcal{B}_{(0)} \mapsto \eta(B) \in \underline{\mathrm{maps}}(\Theta^{(1)}, \mathcal{C})_{(0)} \simeq \mathcal{C}_{(1)}$$
$$b \in \mathcal{B}_{(1)} \mapsto \eta(b) \in \underline{\mathrm{maps}}(\Theta^{(1)}, \mathcal{C})_{(1)} \simeq \mathrm{maps}(\Theta^{(1)} \times \Theta^{(1)}, \mathcal{C})$$

This gives *strong* transformations because

$\Theta^{(1)} \times \Theta^{(1)} =$ .

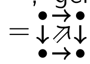
Strategy: Replace $\underline{\mathrm{maps}}(\Theta^{(1)}, \mathcal{C})$ by an (∞, n)-category with the same objects (=1-morphisms in $\mathcal{C}$), but with 1-morphisms = diagrams of shape  in $\mathcal{C}$.

This shape is not symmetric. We want it to be a 1-morphism between arrows in $\mathcal{C}$, so that a "transformation" assigns an arrow to each object and one of these squares to each morphism. So let's decide: in "$\mathcal{C}^{\to}$," the objects are "horizontal" arrows, and a 1-morphism is a box reading down, with source and target (of the 1-morphism) the "vertical" source and target s_v, t_v. In "$\mathcal{C}^{\downarrow}$," the objects are "vertical" arrows; the source and target of a box-as-1-morphism are the "horizontal" source and target s_h, t_h.

Note that then there are two maps $s_v, t_v : \mathcal{C}^{\downarrow} \rightrightarrows \mathcal{C}$ that take any vertical arrow or any diagram just to its top or bottom parts. Similarly $s_h, t_h : \mathcal{C}^{\to} \rightrightarrows \mathcal{C}$.

Defn: Suppose we have defined such $\mathcal{C}^{\to}$ and $\mathcal{C}^{\downarrow}$. A *lax transformation* $\eta : F \Rightarrow G$ between $F, G : \mathcal{B} \rightrightarrows \mathcal{C}$ is a functor $\eta : \mathcal{B} \to \mathcal{C}^{\downarrow}$ with $s_v \circ \eta = F$ and $t_v \circ \eta = G$. An *oplax transformation* is $\eta : \mathcal{B} \to \mathcal{C}^{\to}$ (c.f. §6).

In fact, $\mathcal{C}^{\to}$ and $\mathcal{C}^{\downarrow}$ are the horizontal and vertical 1-morphism categories of a *double* (∞, n)*-category* $\mathcal{C}^{\square}$, i.e. $\mathcal{C}^{\to}_{\bullet} = \mathcal{C}^{\square}_{(1);\bullet}$ and $\mathcal{C}^{\downarrow}_{\bullet} = \mathcal{C}^{\square}_{\bullet;(1)}$. Basic data of an (∞, n)-category $\mathcal{C}$ are spaces $\mathcal{C}_{(i)}$ of i-morphisms for $i \leq n$. Basic data of $\mathcal{C}^{\square}$ are spaces $\mathcal{C}^{\square}_{(i);(j)}$ for $i, j \leq n$.

Strategy: We will define $\mathcal{C}^{\square}_{(i);(j)}$ to be the space of diagrams in $\mathcal{C}$ of shape $\Theta^{(i);(j)}$, generalizing $\Theta^{(k);(0)} = \Theta^{(0);(k)} = \Theta^{(k)}$ and $\Theta^{(1);(1)} =$  (c.f. §5).

Thm (JF–S): $\exists$ unique sequence of diagrams $\Theta^{(i);(j)}$, the *walking (op)lax i-by-j morphism*, with combinatorics like $\Theta^{(i)} \times \Theta^{(j)}$ but no invertible cells, s.t.:

- There are inclusions $s_h, t_h : \Theta^{(i-1);(j)} \rightrightarrows \Theta^{(i);(j)}$ and $s_v, t_v : \Theta^{(i);(j-1)} \rightrightarrows \Theta^{(i);(j)}$ called "horizontal source and target" and "vertical source and target".
- The source of the unique $(i + j)$-dimensional cell $\theta_{i;j}$ is $s_v(\theta_{i;j-1}) \circ s_h(\theta_{i-1;j})$, if i is even, or $t_v(\theta_{i;j-1}) \circ s_h(\theta_{i-1;j})$, if i is odd. (Exactly one of these compositions makes sense.) The target is either $t_h(\theta_{i-1;j}) \circ t_v(\theta_{i;j-1})$ or $t_h(\theta_{i-1;j}) \circ s_v(\theta_{i;j-1})$.

Thm (JF–S): The spaces $\mathcal{C}^{\square}_{(i);(j)} = \mathrm{maps}(\Theta^{(i);(j)}, \mathcal{C})$ do in fact package into a double (∞, n)-category.

The nontrivial part is to prove completeness.

4. Twisted field theory

Thm (JF–S): If $\mathcal{C}$ is symmetric monoidal, so is $\mathcal{C}^{\square}$.

Defn: Choose $\mathcal{C}$ a sym $\otimes$ $(\infty, n+1)$-category, e.g. $\mathcal{C} = \mathrm{VECT}_{n+1}$. Choose sym $\otimes$ (∞, n)-category SPACETIMES controlling n-dim field theory.

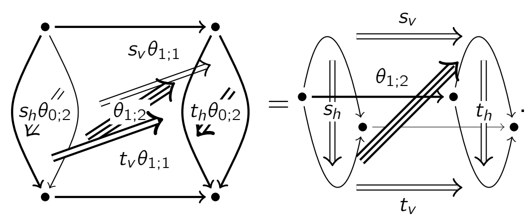
Given $\mathcal{Z} : \mathrm{SPACETIMES} \to \mathcal{C}$, a *lax* $\mathcal{Z}$*-twisted field theory* is a sym monoidal functor $Z : \mathrm{SPACETIMES} \to \mathcal{C}^{\downarrow}$ with $s_v \circ Z = \mathbb{1}$ and $t_v \circ Z = \mathcal{Z}$. For *oplax*, use $\mathcal{C}^{\to}$.

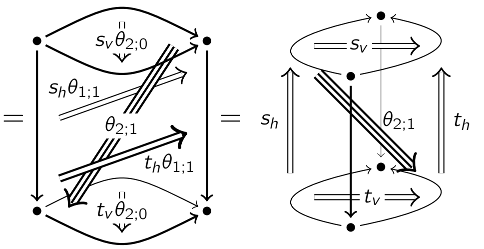
Thm (JF–S): Lax $\mathbb{1}$-twisted field theories are canonically equivalent to "Schrödinger picture" field theories valued in $\Omega\mathcal{C}$ = sym $\otimes$ (∞, n)-category of endomorphisms of $\mathbb{1} \in \mathcal{C}$. (If $\mathcal{C} = \mathrm{VECT}_{n+1}$, then $\Omega\mathcal{C} = \mathrm{VECT}_n$.)

Not true for "lax" $\rightsquigarrow$ "oplax" or "strong."

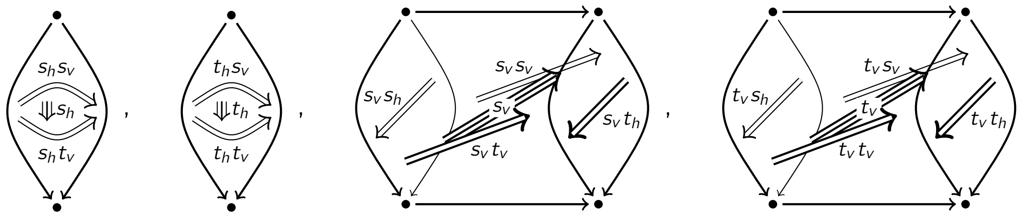
Remark: Oplax (but not lax) $\mathcal{Z}$-twisted field theories are "valued in $\mathcal{Z}$" (c.f. §6). Your choice: is "valued in $\mathcal{Z}$" or "trivially-twisted = untwisted" more important?

5. Examples of walking morphisms

$\Theta^{(1);(2)} =$ 

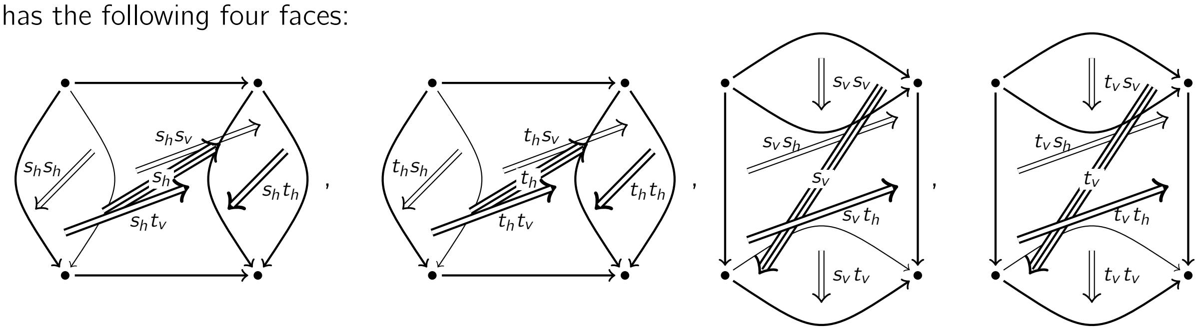
$\Theta^{(2);(1)} =$ 

$\Theta^{(1);(3)}$ has the following four faces:

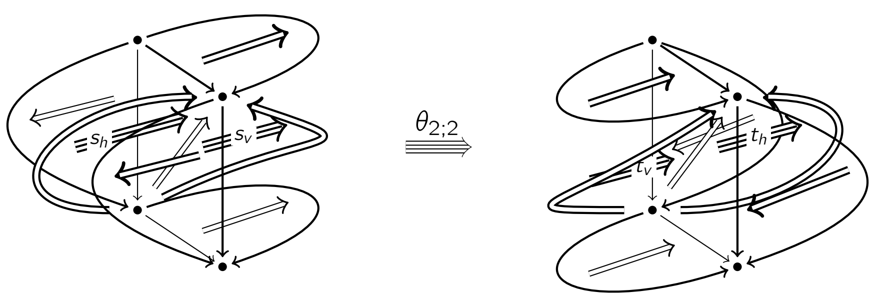


These are glued together along $t_vt_v = s_vt_v$, $t_vs_v = s_vs_v$, $s_vs_h = s_hs_v$, $t_vs_h = s_ht_v$, $s_vt_h = t_hs_v$, and $t_vt_h = t_ht_v$. There is a 4-morphism $\theta_{1;3} : t_v \circ s_h \Rrightarrow t_h \circ s_v$.

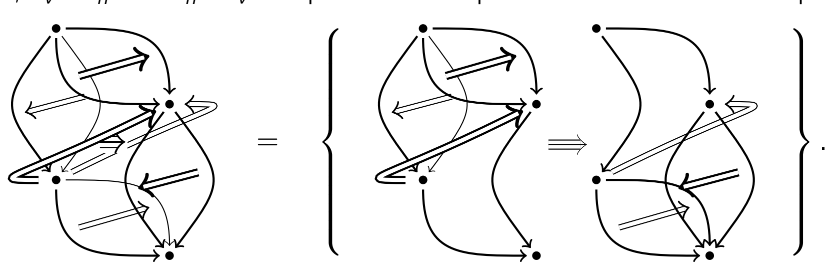
$\Theta^{(2);(2)}$ has the following four faces:



These are glued along $s_hs_h = t_hs_h$, $s_ht_h = t_ht_h$, $s_vs_v = t_vs_v$, $s_vt_v = t_vt_v$, $s_hs_v = s_vs_h$, $s_ht_v = t_vs_h$, $t_hs_v = s_vt_h$, and $t_ht_v = t_vt_h$. Completing $\Theta^{(2);(2)}$ is a 4-morphism $\theta_{2;2} : s_v \circ s_h \Rrightarrow t_h \circ t_v$ filling in the 4-ball with boundary this glued-up 3-sphere:

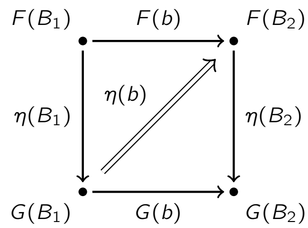
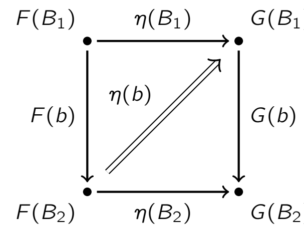
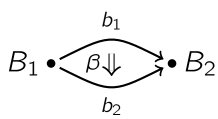
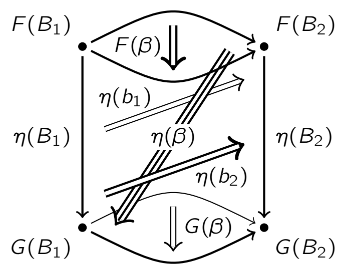
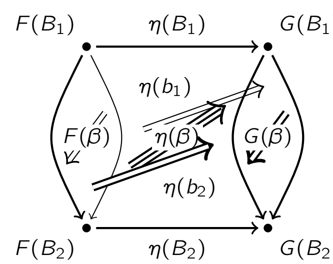


In spite of the distortions, $s_v \circ s_h$ and $t_h \circ t_v$ are parallel 3-morphisms — both are of shape

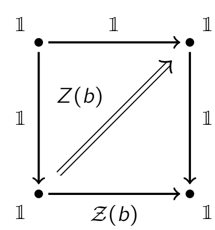
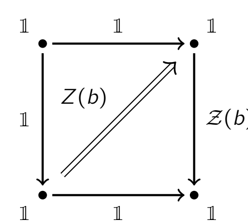
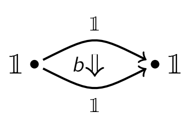
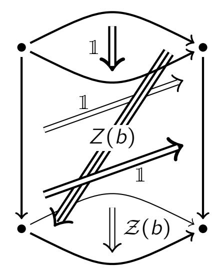
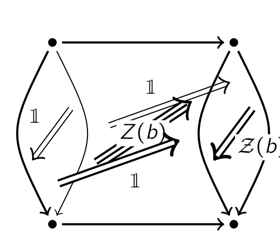


6. Summary: lax versus oplax transformations; lax versus oplax twisted field theory

Lax and oplax transformations $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{B} \rightrightarrows \mathcal{C}$ make the following assignments:

		lax	oplax
object	$B \in \mathcal{B}$	$\eta(B) : F(B) \to G(B)$	$\eta(B) : F(B) \to G(B)$
1-morphism	$\{B_1 \xrightarrow{b} B_2\} \in \mathcal{B}_{(1)}$		
2-morphism			

Therefore a lax/oplax twisted qft $Z : \mathbb{1} \to \mathcal{Z}$ assigns to closed manifolds:

		lax	oplax
object	$b \in \text{SPACETIMES}_{(0)}$	$Z(b) : \mathbb{1} \to \mathcal{Z}(b)$	$Z(b) : \mathbb{1} \to \mathcal{Z}(b)$
1-morphism	$\mathbb{1} \xrightarrow{b} \mathbb{1} \in \text{SPACETIMES}_{(1)}$	 $Z(b) : \mathcal{Z}(b) \Rightarrow \mathbb{1}$	 $Z(b) : \mathbb{1} \Rightarrow \mathcal{Z}(b)$
2-morphism	 $\in \text{SPACETIMES}_{(2)}$	 $Z(b) : \mathbb{1} \Rrightarrow \mathcal{Z}(b)$	 $Z(b) : \mathbb{1} \Rrightarrow \mathcal{Z}(b)$