This talk follows `arXiv:1308.3423`.

Always $\mathrm{H}_\bullet$, etc., is with $\mathbb{Q}$ coeffs.

0. Warm-up: Massey products

For any space M, $\mathrm{H}^\bullet(M)$ is dg com algebra. (Differential is trivial). Why? M is coalgebra: $\mathrm{diag} : M \hookrightarrow M \times M$. $\mathrm{H}^\bullet$ is functorial. And $\mathrm{H}^\bullet(X \times Y) = \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(Y)$.

Question: Does this Com structure come from something on cochain complex $\mathrm{C}^\bullet(M)$?

Exercise: If M is cell complex, and $\mathrm{C}^\bullet$ = cellular cochains, then $\mathrm{C}^\bullet(M)$ is not a Com algebra. E.g. associativity fails. But holds mod exact terms.

Defn: A *homotopy commutative algebra* is a complex (C, ∂) with symmetric operation $m : C \otimes C \to C$ not associative, but $m \circ (\mathrm{id} \otimes m) - m \circ (m \otimes \mathrm{id}) = [\partial, m_3]$ for $m_3 : C \otimes C \otimes C \to C$, satisfying its own axioms and higher coherent homotopies. (Precise defn to follow.)

Wrong answer to Question: Homotopy perturbation theory $\Rightarrow$ given deformation retraction $C \simeq H$ as complexes, and one is h Com algebra, then get formulas for h Com structure on other. Therefore, choose splitting $\mathrm{C}^\bullet(M) \simeq \mathrm{H}^\bullet(M)$, and transfer Com structure on $\mathrm{H}^\bullet(M)$ to h Com structure on $\mathrm{C}^\bullet(M)$.

Why wrong? Resulting operations have no geometric meaning, and give no new topological data about M.

Correct answer to Question: Impose some locality constraint. Then there is unique (in homotopy sense, i.e. contractible space of) h Com structures on $\mathrm{C}^\bullet(M)$ inducing Com structure on $\mathrm{H}^\bullet(M)$. Now use HPT to transfer back. Get h Com structure on $\mathrm{H}^\bullet(M)$, which starts with Com structure but has more data.

Defn: The higher operations in the induced h Com structure on $\mathrm{H}^\bullet(M)$ are the *Massey products*.

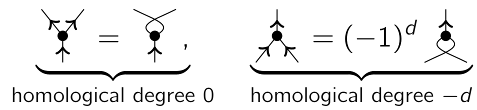
1. Main question for this talk

If M is oriented d-dim manifold, and $\mathrm{C}_\bullet(M)$ = smooth compactly-supported de Rham forms, then $\mathrm{C}_\bullet(M)$ has com multiplication of degree $-d$, inducing *cup product* on $\mathrm{H}_\bullet(M)$. If $\mathrm{C}_\bullet(M)$ = distributional de Rham forms, then has cocom comult. On homology, these satisfy a Frobenius axiom. Can you lift this to chain level?

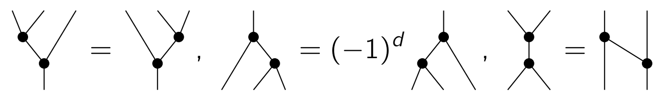
Answer: I will answer "NO" for $M = \mathbb{R}$.

Defn: A *d-shifted com open Frobenius algebra* V, i.e. Frob_d algebra, has *multiplication* $V \otimes V \to V$ of degree $-d$, and *comultiplication* $V \to V \otimes V$ of degree 0, satisfying com and Frob axioms. Open = non(co)unital.

In diagrams (read bottom to top), operations are:



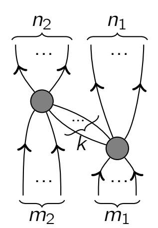
Axioms:



Implicit in the question: These should be weakened to homotopies. But those homotopies should have coherent higher homotopies. How to organize this?

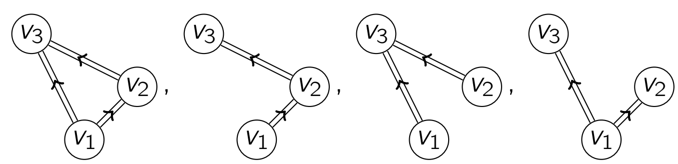
2. Precisifying the problem with properads

Defn: *Props* and *properads* are like operads, but for many-to-many operations: a system of chain complexes $P(m, n)$ of "m-to-n operations," with compositions (and actions of symmetric group $\mathbb{S}_m^{\mathrm{op}} \otimes \mathbb{S}_n$, for permuting inputs/outputs). In a prop, compositions can follow any acyclic directed graph. In a properad, compositions must be connected. So are generated by:



$$: \begin{array}{l} P(m_1, k + n_1) \otimes P(m_2 + k, n_2) \\ \quad \to P(m_1 + m_2, n_1 + n_2) \end{array}$$

for $k \geq 1$. Four associativity axioms, for diagrams like:



Theorem (Vallette): Both props and properads have model category structures for which fibrations = surjections and weak equivalences = quasiisomorphisms. The free functor from properads to props is exact.

Defn: For P a prop(erad), an h P algebra is an algebra for a cofibrant resolution of P. **Cor of theorem:** It doesn't matter which h P. If P is properad, computing h P in properads = computing it in props.

Warning: Another way to describe Frob algebras is with dioperads, which have only tree-like composition. Free functor to props is *not* exact. So gives different meaning of "homotopy algebra."

3. Technical tool: Koszul duality

How to compute cofibrant resolutions?

Let T = some labels on vertices, with prescribed symmetry action. (I.e. T is $\mathbb{S}$-bimodule.) Then T freely generates a properad $\mathcal{F}(T)$, which is also a coproperad (meaning has decompositions). Let $\mathcal{F}^{(k)}(T)$ be part of $\mathcal{F}(T)$ with weight k under $\mathbb{Q}^\times$ action on T.

Given properad P, consider $\mathcal{F}(P[1])$. The degree-(-1) map $\sum_k$(binary composition with k internal strands) : $\mathcal{F}^{(2)}(P[1]) \to \mathcal{F}^{(1)}(P)$ extends uniquely to a coderivation ∂ of coproperad $\mathcal{F}(P[1])$. Associativity $\Leftrightarrow \partial^2 = 0$.

Defn: $\mathbf{B}P = \big(\mathcal{F}(P[1]), \partial\big)$ is the *bar complex* of P.

Defn: Conversely, given coproperad Q, under mild conditions $\sum$(binary decomps) : $\mathcal{F}^{(1)}(Q[-1]) \to \mathcal{F}^{(2)}(Q[-1])$ converges, extends to derivation ∂ of properad $\mathcal{F}(Q[-1])$. $\mathbf{B}^*Q = \big(\mathcal{F}(Q[-1]), \partial\big)$ is the *cobar complex* of Q.

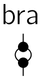
Fact: $\mathbf{B}^*\mathbf{B}P \overset{\sim}{\to} P$ is (usually) a cofibrant resolution.

Problem: It's big. $\mathbf{B}^*Q$ is cofibrant. Can shrink $\mathbf{B}P$?

Defn: A *quadratic properad* is $\mathcal{F}(T)/R$ with $R \subseteq \mathcal{F}^{(2)}(T)$.

E.g.: Frob_d is quadratic.

Defn: *Quadratic dual* $P^{¡}$ of $P = \mathcal{F}(T)/R$ is max subcoproperad of $\mathcal{F}(T[1])$ such that $P^{¡} \cap \mathcal{F}^{(2)}(T[1]) = R[2]$.

E.g.: $(\mathrm{Frob}_d)^{¡}$ is linear dual of properad controlling involutive Lie bialgebras with deg(bracket) $= d - 1$ and deg(cobracket) $= -1$. *Involutive*:  $= 0$

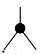
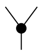
Exercise: There is canonical fibration $\mathbf{B}^*P^{¡} \twoheadrightarrow P$.

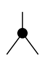
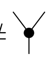
Defn: P is *Koszul* if this is acyclic.

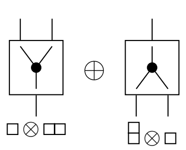
Fact: When d odd, Frob_d is Koszul. (Even case is open.)

Cor: Can use $\mathrm{h\,Frob}_1 = \mathbf{B}^*(\mathrm{Frob}_1^{¡})$ when studying main question.

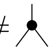
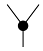
4. Details on $\mathbf{B}^*(\mathrm{Frob}_1^{¡})$

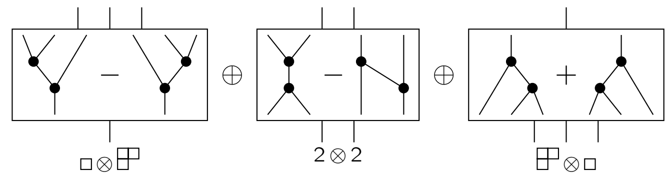
Frob_1 is graded by (# , # ), hence so is $\mathbf{B}^*(\mathrm{Frob}_1^{¡})$. Generators of $\mathbf{B}^*(\mathrm{Frob}_1^{¡})$ are labeled by elements of $\mathrm{Frob}_1^{¡}$.

Generators of $\mathbf{B}^*(\mathrm{Frob}_1^{¡})$ with #  +#  = 1 are:

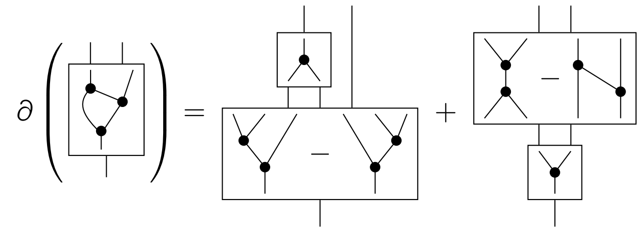


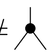
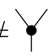
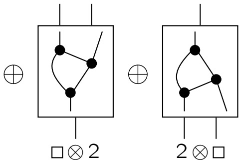
Young diagrams label reps of $\mathbb{S}^{\mathrm{op}} \times \mathbb{S}$.

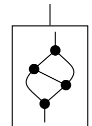
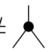
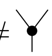
The #  + #  = 2 piece:



To test if something in $\mathcal{F}(T[1])[-1]$ is generator, test if its derivative in $\mathbf{B}^*(\mathcal{F}(T[1]))$ lands in $\mathbf{B}^*(\mathrm{Frob}_1^{¡})$. E.g.:

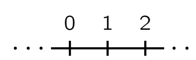


⇒#  +#  = 3 part is (trees) ⊕ .

Similarly,  occurs when #  + #  = 4.

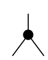
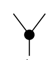
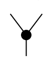
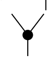
5. Nonexistence of quasilocal $\mathrm{h\,Frob}_1$ actions

Main question, redux: Does $\mathrm{h\,Frob}_1$ act on $C_\bullet(\mathbb{R})$ such that each operation moves chains only finitely much? Use cellular chains:

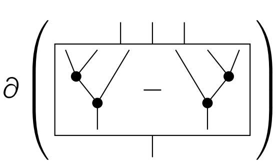


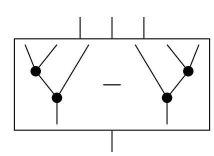
Any operation $C_\bullet(\mathbb{R})^{\otimes m} \to C_\bullet(\mathbb{R}^n)$ has "matrix" in $\mathbb{R}^{m+n}$, and I want it supported in a finite-radius nbhd of diagonal.

Let's try to represent $\mathbf{B}^*(\mathrm{Frob}_1^{¡})$ inductively.

Representing  **and** **:** Choose Thom forms. E.g.  (pt) = pt ⊗ pt and  (interval $[a, a+1]$) $= \frac{1}{2}\big(([a] + [a+1]) \otimes [a, a+1] + [a, a+1] \otimes ([a] + [a+1])\big)$. Space of choices is contractible.

Basic fact of obstruction theory: If you change a choice to something homotopic, you don't change whether later choices can be made.

Second step: ∂  is already determined:

it is a difference of Thom forms. Hence 

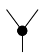
can be chosen. Need to make sure to choose it in the 2-dim rep of $\mathbb{S}_3$ — can do this by averaging.

One answer that works is pt $\mapsto 0$ and for intervals,

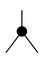
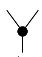
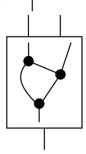
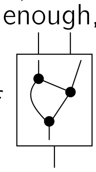
$$\mapsto \frac{1}{6}[a, a+1] \otimes ([a+1]-[a]) \otimes [a, b] + \frac{1}{12}\big(([a+1]-[a]) \otimes [a, a+1] \otimes [a, a+1] + [a, a+1] \otimes [a, a+1] \otimes ([a+1]-[a])\big)$$

Similar for other terms.

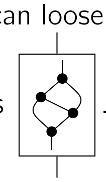
Third step: For every generator γ, $\partial\gamma$ is already determined. And $\partial(\partial\gamma) = 0$ by associativity for already-determined pieces.

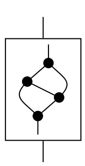
Lemma: Space of quasilocal m-to-n operations has homology only in degree $-m+1$. Generator of $\mathbf{B}^*(\mathrm{Frob}_1^{\mathrm{i}})$ has degree #  $-1 = n-2+$genus. So deg ∂(generator) $= -m+1$ only when $m+n+$genus $=4$. Otherwise, ∂(generator) is automatically exact.

Lemma: There are nonhomotopic choices only when $m+n+$genus $=3$, i.e. never.

Cor: At #  + #  = 3, only thing to check is whether we can represent . We can calculate the action of its derivative. Sure enough, it's exact. Choose an explicit choice for action of , e.g.:

$$[a, a+1] \mapsto -\frac{1}{12}[a, a+1] \otimes [a, a+1] \text{ works.}$$

Fourth step: If we can complete the fourth step, we win, since the only times we can loose are $m+n+$genus $=4$, so last chance to loose is . But disaster strikes!

$\partial\Big($  $\Big)$ acts by (something homotopic to) $-\frac{1}{12}$.

And id_V is exact in End(V) iff V is acyclic. Since $\mathrm{C}_\bullet(\mathbb{R})$ has homology, $-\frac{1}{12}$ is not exact.