Functorial axioms for Heisenberg-picture qft, Theo Johnson-Freyd, 12 January 2015. p 1/2.

1. Schrödinger v. Heisenberg quantum mechanics

Schrödinger	textbook Heisenberg	improved Heisenberg	Schrödinger $\to$ Heisenburg functor "End"
"Hilbert" space V	assoc algebra A	assoc algebra A	$A = \mathrm{End}(V)$ (e.g. bounded operators)
$u_t : V \to V$, $t \in \mathbb{R}_{>0}$ s.t. $u_{t_1+t_2} = u_{t_1} u_{t_2}$	autos $f_t : A \overset{\sim}{\to} A$ s.t. $f_{t_1+t_2} = f_{t_1} f_{t_2}$	pointed bimodules s.t. group law	${}_A A_A$ with left action twisted by $a \mapsto u_t a u_1^{-1}$ $\cong$ usual ${}_A A_A$ but pointed by u_t.
distinguished $v_i \in V$, $w_j \in V^*$, $a_k : V \to V$	???	distinguished pointed (bi)modules	${}_A V$ pointed by v_i, $(V^*)_A$ pointed by w_j, ${}_A A_A$ pointed by a_k

2. Non-affine quantum spaces

Defn: $\mathrm{Vect}_0 =$ ground ring. $\mathrm{Vect}_1 = \mathrm{Vect}$ or DGVect. $\mathrm{Vect}_2 =$ appropriate 2-category of linear categories.

E.g.: I like strict categories. For me $\mathrm{Vect}_0 = \mathbb{K}$, $\mathrm{Vect}_1 = \mathrm{Mod}_{\mathbb{K}}$, and $\mathrm{Vect}_2 = \mathrm{Pres}_{\mathbb{K}} =$ the bicategory of $\mathbb{K}$-linear locally presentable categories with 1-morphisms $=$ $\mathbb{K}$-linear cocontinuous functors.

Defn: Let X be a "space" (scheme or manifold or ...). $\mathcal{O}_k(X) =$ symmetric-monoidal k-category of functions $X \to \mathrm{Vect}_k$. **E.g.:** $\mathcal{O}_0(X) = \mathcal{O}(X) =$ usual algebra of functions. $\mathcal{O}_1(X) = \mathrm{Qcoh}(X) =$ (derived?) category of quasicoh sheaves of $\mathcal{O}$-modules.

Rmk: $\mathcal{O}_{k-1}(X) = \mathrm{End}_{\mathcal{O}_k(X)}(\text{unit object})$.

Defn: X is *k-affine* if $X \to \mathrm{Spec}(\mathcal{O}_k(X))$ is an equiv. **Rmk:** "$\mathrm{Spec}(\mathcal{C})$" is a stack in fpqc topology.

Gelfand–Naimark theorem: Most spaces from functional analysis and point-set topology are 0-affine.

Grothendieck: Most spaces from algebraic geometry are not 0-affine.

Tannakian philosophy: Most spaces from algebraic geometry are 1-affine.

E.g.: qcqs schemes are 1-affine (Brandenburg–Chirvasitu), as are affine ind-schemes (Brandenburg–Chirvasitu–JF) and Nötherian algebraic stacks with affine stabilizers (Hall–Rydh). Algebraic stacks with non-affine stabilizers are not 1-affine (Hall–Rydh).

Defn: A *1-quantum space* does is not have a commutative algebra of functions, but rather an *associative* algebra. More generally, *n-quantum spaces* have algebras of functions that are *n-algebras*, i.e. algebras for the operad of little disks in $\mathbb{R}^n$. Spec(n-algebra) is a *0-affine n-quantum space*. ∞-quantum $=$ commutative $=$ classical.

E.g.: A (0-affine) *0-quantum space* is a pointed $\mathbb{K}$-module (object in Vect_1).

Associative algebras (0-affine 1-quantum spaces) do not have symmetric monoidal categories of modules. But Mod_A is a *pointed category*, pointed by A_A.

Defn: A *1-affine 1-quantum space* is a pointed object in Vect_2 (i.e. pointed category with good properties).

More generally, if $k \leq n$, a *k-affine n-quantum space* is an $(n-k)$-algebra object in Vect_{k+1}.

E.g.: (Classifying stacks of) quantum groups are 1-affine 3-quantum. (Braided monoidal category $=$ 2-algebra in Vect_2.)

Bicategory of 1-affine 1-quantum spaces: Objects are pointed categories $(\mathcal{C}, C)$ (linear, locally presentable, etc.). 1-morphisms $(\mathcal{C}, C) \to (\mathcal{D}, D)$ are "oplax pointed functors" (F, f) with $F : \mathcal{C} \to \mathcal{D}$ and $f : D \to F(C)$. 2-morphisms are natural transformations making appropriate triangle commute.

This is called "$\mathrm{Alg}_0(\mathrm{Vect}_2)$."

C.f.: $\mathrm{Alg}_1(\mathrm{Vect}_1)$: objects are associative algebras, 1-morphisms are pointed bimodules, 2-morphisms are pointed intertwiners. Related by functor "Mod":

$A \mapsto (\mathrm{Mod}_A, A_A)$.

$({}_A M_B, m) \mapsto ((-) \otimes_A M, m : B_B \to M_B)$.

Mod $\circ$ End :

$V \mapsto (\mathrm{Mod}_{\mathrm{End}(V)}, \mathrm{End}(V)_{\mathrm{End}(V)}) \cong (\mathrm{Vect}_1, V)$.

$(f : V \to W) \mapsto \mathrm{Mod}({}_{\mathrm{End}(V)} \hom(V, W)_{\mathrm{End}(W)}, f)$

$\cong (\mathrm{id}_{\mathrm{Vect}_1}, f : V \to W)$.

3. Functorial definition of quantum field theory

Different types of qft require different geometries on spacetime. Fix some geometry γ. $\mathrm{Bord}^{\gamma}_{d-n,\dots,d} = n$-category of "bordisms of dimensions $d - n \leq \dim \leq d$ equipped with γ-geometry." E.g. Lurie's theorem (cobordism hypothesis) is about $\mathrm{Bord}^{d\text{-framing}}_{0,\dots,d}$.

Defn (Atiyah, Segal, etc.): An *n-extended d-dimensional Schrödinger-picture qft (for geometry γ)* is a symmetric monoidal functor $\mathrm{Bord}^{\gamma}_{d-n,\dots,d} \to \mathrm{Vect}_n$.

Defn: An *n-extended d-dimensional k-affine Heisenberg-picture qft* is a symmetric monoidal functor $\mathrm{Bord}^{\gamma}_{d-n,\dots,d} \to \mathrm{Alg}_{n-k}(\mathrm{Vect}_{k+1})$.

Here $\mathrm{Alg}_{n-k}(\mathrm{Vect}_{k+1})$ is the $(n+1)$-categry with:

- Objects = $(n-k)$-algebras in Vect_{k+1}
- 1-morphisms = "bimodule" $(n-k-1)$-algebras
- ...
- $(n-k)$-morphisms = pointed "bimodules between ... between bimodules" in Vect_{k+1}
- $(n-k+1)$-morphisms = pointed 1-morphisms in Vect_{k+1} intertwining all actions
- ...

In the early 21st century, everyone knows what a strict 1-category is, but higher categories are still confusing, so it takes theorems to check that they exist:

Theorem (Scheimbauer–Calaque): Let $\mathcal{S}$ be a symmetric monoidal $(\infty, 1)$-category satisfying mild properties. Then there is a symmetric monoidal $(\infty, n-k)$-category $\mathrm{Alg}_{n-k}(\mathcal{S})$ with above 0- through $(n-k)$-morphisms.

Theorem (JF–Scheimbauer): If $\mathcal{S}$ is (∞, k), then $\mathrm{Alg}_{n-k}(\mathcal{S})$ extends to an (∞, n)-category.

4. Important non-examples

Take γ = framing, $n = d$. Recall Lurie's theorem:

Cobordism hypothesis: {Symmetric monoidal functors $\mathrm{Bord}^{n\text{-fr}}_{0,\dots,n} \to \mathcal{C}\} \simeq \{n$-dualizable objects in $\mathcal{C}$}.

Theorem (Scheimbauer–Calaque): Every object of $\mathrm{Alg}_{n-k}(\mathcal{S})$ is $(n-k)$-dualizable.

Theorem (JF–Scheimbauer): Let $\mathcal{S}$ be (∞, k)-category. The only n-dualizable in $\mathrm{Alg}_{n-k}(\mathcal{S})$ is the unit.

So the interesting dualizability is in between.

Exercise: Take $\mathrm{Vect}_2 = \mathrm{Pres}_{\mathbb{K}}$. If $(\mathcal{C}, C) \in \mathrm{Alg}_0(\mathrm{Vect}_2)$ is 1-dualizable, then C is compact projective.

Corollary: Let X be a scheme. $(\mathrm{Qcoh}(X), \mathcal{O}_X) \in \mathrm{Alg}_0(\mathrm{Vect}_2)$ is 1-dualizable iff X is (0-)affine.

It gets worse:

Theorem (Brandenburg–Chirvasitu–JF): Let X be a scheme. If X contains a closed projective subscheme, then $\mathrm{Qcoh}(X)$ is not 1-dualizable in $\mathrm{Pres}_{\mathbb{K}}$.

Theorem (Brandenburg–Chirvasitu–JF): Let C be a coassociative coalgebra. Comod^C is 1-dualizable in $\mathrm{Pres}_{\mathbb{K}}$ iff it has enough projectives.

E.g.: If $C = \mathcal{O}(G)$ for G an affine algebraic group. Has enough projectives when reductive in char 0, but not when solvable, nor when reductive in char p.

Conj: $\mathrm{Qcoh}(X)$ is 1-dualizable in $\mathrm{Pres}_{\mathbb{K}}$ iff X is affine.
Conj: $\mathcal{C}$ is 1-dualizable in $\mathrm{Pres}_{\mathbb{K}}$ iff has enough compact projectives. **Rmk:** Both $\Leftarrow$s are elementary.

Rmk: This nondualizability goes away if you use derived categories, since then everything is projective, so only need enough compacts (Ben-Zvi–Francis–Nadler).

5. Heisenberg-picture Chern–Simons theory

Theorem (JF): $\mathcal{C} \in \mathrm{Alg}_2(\mathrm{Pres}_{\mathbb{K}})$ is 3-dualizable if $1_{\mathcal{C}} \in \mathcal{C}$ is compact projective and $\mathcal{C}$ is generated by its dualizable objects. It is SO(3)-invariant (i.e. defines an oriented tqft rather than a framed one) if it is equipped with a "ribbon structure."

Main e.g.: $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$. $\mathcal{C} = \mathrm{TL}$ = cocompletion of Temperley–Lieb category = rep thy of Temperley–Lieb algebra. Corresponding tqft $M \mapsto \int_M \mathrm{TL}$ packages together all objects of Kauffman-bracket skein theory:

- For M = 3-manifold with boundary, $\int_M \mathrm{TL}$ = Przytycki's *relative skein modules*.
- For M = 2-manifold, $\int_M \mathrm{TL}$ is a pointed category with End(pointing object) = usual skein algebra.
- $\int_{S^1} \mathrm{TL}$ = annular Temperley–Leib.

One power of working with $\mathrm{Pres}_{\mathbb{K}}$ is that all constructions are well-behaved for "tensoring"-type operations. In particular, for any commutative ring hom $\mathbb{Z}[q, q^{-1}] \to R$, there is a category $\mathrm{TL} \otimes_{\mathbb{Z}[q,q^{-1}]} R \in \mathrm{Alg}_2(\mathrm{Pres}_R)$, and $(\int_M \mathrm{TL}) \otimes_{\mathbb{Z}[q,q^{-1}]} R \cong \int_M (\mathrm{TL} \otimes_{\mathbb{Z}[q,q^{-1}]} R)$.

E.g.: $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ via $q \mapsto -1$. Then $\mathrm{TL} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} = \mathrm{Rep}(\mathrm{SL}(2))$.

For $\mathcal{C} = \mathrm{Rep}(G)$ (with trivial module projective, i.e. reductive in char=0), $\int_{\square} \mathcal{C}$ is "classical topological gauge theory":

- For $\dim M \leq 2$, $\int_M \mathrm{Rep}(G) = \mathrm{Qcoh}$(stack of G-local systems on M).
- For $\dim M = 3$, $\int_M \mathrm{Rep}(G) \in \int_{\partial M} \mathrm{Rep}(G)$ is pushforward of $\mathcal{O}_{\mathrm{Loc}_G(M)}$ along restriction map $\mathrm{Loc}_G(M) \to \mathrm{Loc}_G(\partial M)$.

Optimism: This might help answer various "semiclassics" questions from quantum topology, e.g. "AJ conjecture" relating colored Jones polynomial to character variety.

(I could prove AJ if I could split a certain monomorphism. It's approximately the inclusion of holonomic sequences among all sequences.)