Where does the Spin–Statistics Theorem come from? Theo Johnson-Freyd, 23 November 2015.

1. Punchline of the talk

arXiv:1507.06297.

Cat #	alg. closure of $\mathbb{R}$	tangential structure	physical phenomenon
0	$\mathbb{C}$	orientation	unitarity
1	$\mathrm{SuperVect}_{\mathbb{C}}$	spin	spin-statistics

Manifolds $\rightsquigarrow$ certain universal groups
Galois theory $\rightsquigarrow$ certain universal torsors

2. A few words on Quantum Field Theory

In mathematics, $\exists$ various proposed defns of "QFT," e.g. functor on bordism category. In physics, QFTs just *are* — our job is mathematical models. Topological qfts continue to provide insights into mathematics in topology, rep thy, algebraic geo, symplectic geo, functional analysis, category theory, homotopy algebra, ...

All proposed defns of "QFT" package the data as an assignment in which geometric inputs (manifolds with metric, spin structure, . . .) map to algebraic outputs ("S-"matrices, Hilbert spaces of "states", categories of "boundary branes", . . .). *Locality* $\Rightarrow$ QFTs are a quantization of *sheaves*.

3. Orientations and oriented QFT

Consider assignment $\mathrm{Or} : \{\text{manifolds}\} \to \{\text{sets}\}$, $M \mapsto \{\text{orientations on } M\}$. It satisfies:

1. It is not (contravariantly) functorial for smooth functions, but is for *étale maps* (local diffeos).
2. $\mathrm{Or}(M_1 \cup_N M_2) = \mathrm{Or}(M_1) \times_{\mathrm{Or}(N)} \mathrm{Or}(M_2)$.
3. $\mathrm{Or}(f) = \mathrm{Or}(g)$ if $f, g : M_1 \to M_2$ are homotopic.
4. $\mathrm{Or}(M) = \mathrm{Or}(M \times \mathbb{R})$

Lemma: Any assignment $F : \mathrm{Man} \to \mathrm{Set}$ satisfying 1–4 is determined by $F(\{\mathrm{pt}\}) = F(\mathbb{R}^d)$ with its action of $\mathbb{Z}/2 = \{\pm 1\} \subseteq \mathrm{maps}(\mathbb{R}^d, \mathbb{R}^d)$.

Pf: 1 $\Leftrightarrow$ *presheaf* on site $\mathrm{Man}_{\text{ét}}$ of manifolds and étale map. 2 $\Leftrightarrow$ *sheaf* (essentially) on $\mathrm{Man}^d_{\text{ét}}$. Existence of good open covers $\Rightarrow$ F determined by sets $F(\mathbb{R}^d)$ with actions of monoids $\mathrm{maps}_{\text{ét}}(\mathbb{R}^d, \mathbb{R}^d)$. 3 $\Rightarrow$ these actions factor through $\pi_0(\mathrm{maps}_{\text{ét}}(\mathbb{R}^d, \mathbb{R}^d)) = \mathbb{Z}/2$. 4 $\Rightarrow$ $F(\mathbb{R}^d) = F(\{\mathrm{pt}\})$. $\square$

Main observation: What is $\mathbb{Z}/2$-set $\mathrm{Or}(\{\mathrm{pt}\})$? It is the *trivial torsor* $\mathbb{Z}/2$ acting on itself by translation.

For some QFTs Z, values depend on orientation of input manifold: $Z : \mathrm{Man}^{\mathrm{Or}} \to \{\text{possible values}\}$. But "oriented manifold" = "manifold M with point in $\mathrm{Or}(M)$," so

$$\begin{tikzcd} \mathrm{Man}^{\mathrm{Or}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Set}_{\{\mathrm{pt}\}/} \arrow[d] \\ \mathrm{Man} \arrow[r, "\mathrm{Or}"] & \mathrm{Set} \end{tikzcd}$$

Alternately, can think of Z as sending unoriented manifold M to a bundle over $\mathrm{Or}(M)$.

4. Unitary structures and unitary QFTs

In Set, $\mathbb{Z}/2$ has unique torsor. In $\mathrm{Schemes}_{\mathbb{R}}$, $\exists$ canonical nontrivial $\mathbb{Z}/2$-torsor, namely $\mathrm{Spec}(\mathbb{C})$ acted on by complex conjugation. Indeed, $\mathbb{Z}/2 = \mathrm{Gal}^{\mathrm{abs}}(\mathbb{R})$.

Defn: *Unitary structures* are the sheaf $\mathrm{Unitary} : \mathrm{Man}_{\text{ét}} \to \mathrm{Schemes}_{\mathbb{R}}$ satisfying 1–4 above s.t. $\mathrm{Unitary}(\{\mathrm{pt}\}) = \mathrm{Spec}(\mathbb{C})$ as a $\mathbb{Z}/2$-torsor. Explicitly, $\mathrm{Unitary}(M) = \big(\mathrm{Or}(M) \times \mathrm{Spec}(\mathbb{C})\big)/(\mathbb{Z}/2)$.

If you build $\mathrm{Man}^{\mathrm{Unitary}}$ as

$$\begin{tikzcd} \mathrm{Man}^{\mathrm{Unitary}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathrm{Schemes}_{\mathbb{R}})_{\{\mathrm{pt}\}/} \arrow[d] \\ \mathrm{Man} \arrow[r, "\mathrm{Unitary}"] & \mathrm{Schemes}_{\mathbb{R}} \end{tikzcd}$$

in usual-categories, you don't get much, because if $M \neq \emptyset$, $\mathrm{Unitary}(M)$ does not have any ($\mathbb{R}$-)points.

Defn: The category $\mathrm{Man}^{\mathrm{Unitary}}$ of *manifolds with unitary structure* (and étale maps) is the above fiber product interpreted as a category *internal to* $\mathrm{Schemes}_{\mathbb{R}}$. This means that there is an $\mathbb{R}$-scheme of objects, rather than a set, and an $\mathbb{R}$-scheme of morphisms.

Provided {possible values} also makes sense $\mathbb{R}$-algebraically, a *unitary QFT* is a quantum field theory *internal to* $\mathrm{Schemes}_{\mathbb{R}}$ whose inputs range over $\mathrm{Man}^{\mathrm{Unitary}}$. (Actually, use $\mathrm{Stacks}_{\mathbb{R}}$ instead of $\mathrm{Schemes}_{\mathbb{R}}$.)

Lemma: Equivalently, a unitary QFT assigns to each $M \in \mathrm{Man}$ a bundle over $\mathrm{Unitary}(M)$. $\square$

E.g.: Suppose that a plain QFT is supposed to assign a (real) vector space to M. Then a unitary QFT should assign a bundle of vector spaces over $\mathrm{Unitary}(M)$, because the internal category $\mathrm{Vect}_{\mathbb{R}}$ *is* the stack QCoh.

So for M connected orientable, get a bundle over $\mathrm{Unitary}(M) \cong \mathrm{Spec}(\mathbb{C})$, i.e. a *complex* vector space. But not really: in general, $\nexists$ canonical iso $\mathrm{Spec}(\mathbb{C}) \cong \mathrm{Unitary}(M)$ — you can't decide whether to complex conjugate. Really get a complex vector space for each orientation, and orientation reversal = complex conj.

5. Spin structures and spin QFT

Defn: A *spin structure* on $M \in \mathrm{MAN}^d$ is a principal $\mathrm{Spin}(d)$ bundle $P \to M$ with iso $P \times_{\mathrm{Spin}(d)} \mathrm{GL}(d,\mathbb{R}) \cong \mathrm{Frame}(M)$. Collection of spin structures $\mathrm{Spin}(M)$ is not a set: it is a *groupoid*.

$\mathrm{Spin} : \mathrm{MAN} \to \mathrm{GPOID}$ satisfies 1–4 above (1+2 $\rightsquigarrow$ *stack of groupoids*).

Lemma: A stack of groupoids F satisfying 1–4 is determined by groupoid $F(\{\mathrm{pt}\})$ with action of *fundamental groupoid* $\pi_{\leq 1}(\mathrm{maps}_{\text{ét}}(\mathbb{R}^d, \mathbb{R}^d)) = \mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$.

Pf: Repeat earlier proof. Calculation $\pi_{\leq 1} = \mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$ follows from [Bott 1959]. $\square$

Rmk: For ∞-stacks satisfying 1–4, need full homotopy type of $\bigcup_{d\to\infty} \mathrm{maps}_{\text{ét}}(\mathbb{R}^d, \mathbb{R}^d) \simeq \mathrm{O}(\infty)$. For 1–3, use $\mathrm{maps}_{\text{ét}}(\mathbb{R}^d, \mathbb{R}^d) \simeq \mathrm{O}(d)$. This is a version of *cobordism hypothesis*. **Exercise:** Directly construct the ∞-stack corresponding to a given $\mathrm{O}(d)$- or $\mathrm{O}(\infty)$-action.

Main observation: $\mathrm{Spin}(\{\mathrm{pt}\})$ is trivial $\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$ torsor. **Defn:** $\mathrm{MAN}^{\mathrm{Spin}}$, Spin QFT are as above.

6. Categorified schemes

We could look for interesting $\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$-torsors among stacks over $\mathbb{R}$ in the usual sense, but there aren't any, because $\mathrm{Spec}(\mathbb{C})$ is étale contractible. But this is just a failure of sufficient categorification.

Defn: A *categorified commutative $\mathbb{R}$-algebra* is an $\mathbb{R}$-linear sym mon category (+ abelian, etc.).

E.g.: $(\mathrm{MOD}_A, \otimes_A)$ for A commutative over $\mathbb{R}$.

Thm (Hall–Rydh): Noetherian stack X with affine stabilizers is functorially recoverable from its cat com $\mathbb{R}$-alg $\mathrm{QCOH}(X)$. **Thm (Brandenburg–Chirvasitu–JF):** Ditto affine ind-schemes. **Slogan:** "Categorified affine."

Defn: Category of *categorified affine schemes* is opposite that of cat com algebras. $\mathrm{Spec}(A) = \mathrm{Spec}(\mathrm{MOD}_A)$.

In joint work with Chirvasitu and Elmanto, we are developing the "étale site" of cat affine schemes. Then will have "categorified stacks" as stacks for this site.

Main example: Cat com $\mathbb{R}$-algerba $\mathrm{SUPERVECT}_{\mathbb{C}}$ is monoidally, but not symmetric monoidally, equivalent to $\mathrm{REP}_{\mathbb{C}}(\mathbb{Z}/2)$. Difference is in symmetry (sign) $\otimes$ (sign) $\to$ (sign) $\otimes$ (sign); in $\mathrm{SUPERVECT}_{\mathbb{C}}$, is -1. $\mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{C}})$ has no $\mathbb{C}$-points.

7. Spin–statistics QFTs

Defn: A *categorified field* $\mathcal{K}$ is a cat com alg s.t. for $\mathcal{A} \neq 0$, any map $\mathcal{K} \to \mathcal{A}$ is injective (on morphisms and on iso classes of objects). $\mathcal{K}$ is *algebraically closed* if for $\mathcal{A} \neq 0$ *and finite-dimensional*, and map $\mathcal{K} \to \mathcal{A}$ splits. I will not repeat here the defn of "finite dimensional".

Thm (essentially Deligne 2002): Categorified algebraic closure of $\mathbb{R}$ is $\mathrm{SUPERVECT}_{\mathbb{C}}$.

Galois group is $\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts as complex conjugation and $\mathrm{B}(\mathbb{Z}/2)$ by "$(-1)^f$", the "fermion number" count. Thus in $\mathrm{CATSCHEMES}_{\mathbb{R}}$, $\exists$ canonical nontriv $\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$-torsor, namely $\mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{C}})$.

Defn: Let SpinStats denote stack on $\mathrm{MAN}_{\text{ét}}$ valued in categorified affine schemes satisfying 1–4 s.t. $\mathrm{SpinStats}(\{\mathrm{pt}\}) = \mathrm{Spec}(\mathrm{SUPERVECT})$ with this $\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$-action.

$$\mathrm{SpinStats}(M) = \frac{\mathrm{Spin}(M) \times \mathrm{Spec}(\mathrm{SUPERVECT})}{\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)}.$$

Set $\mathrm{MAN}_{\text{ét}}^{\mathrm{SpinStats}}$ and "SpinStats QFT" as above.

These are notions internal to {categorified stacks}: their objects and morphisms can so on can vary with a parameter ranging over, say, $\mathrm{Spec}(\mathrm{SUPERVECT})$.

E.g.: Suppose a usual QFT would assign to M a vector space. Then a SpinStats QFT assigns to M a bundle of vector spaces over $\mathrm{SpinStats}(M)$. Note: a bundle of (real!) vector spaces over $\mathrm{Spec}(\mathrm{SUPERVECT})$ is a *complex super* vector space, by definition.

More precisely, for each spin structure, get a complex supervector space. $\mathbb{Z}/2$-equivariance $\Rightarrow$ complex conj = orientation reversal, just as before. $\mathrm{B}(\mathbb{Z}/2)$-equivariance $\Rightarrow$ $(-1)^f =$. What I mean: for any spin manifold, there is an interesting spin diffeomorphism which is the identity on underlying manifolds but in which the spin structure "twists around by 360° along the diffeomorphism"; this diffeo should act on the value of the QFT by $(-1)^f$.

8. Questions and extensions

What about $\mathbb{F}_p$? Expect $\mathrm{SUPERVECT}_{\bar{\mathbb{F}}_p}$ to be *separable* closure, but Ostrik has constructed a *totally inseparable* extension of $\mathrm{SUPERVECT}_{\bar{\mathbb{F}}_p}$ called VER_p.

What about higher categories? Expect an "∞-categorified abs Galois group of $\mathbb{R}$." Perhaps it is $\mathrm{O}(\infty)$.

What about positivity? Expect a story starting with $\mathbb{R}_{\geq 0}$ in place of $\mathbb{R}$; will need "algebraic geometry over $\mathbb{F}_1$."