Talk based on `arXiv:1507.06297`.

0. Main message: sharp analogy between...

$\mathbb{C}$	Orientations	Unitarity
SUPERVECT	Spin structures	Spin-Statistics

1. Idea of structured field theory

Full definition of "quantum field theory" is still open, but consensus is that the following captures some (e.g. topological) examples:

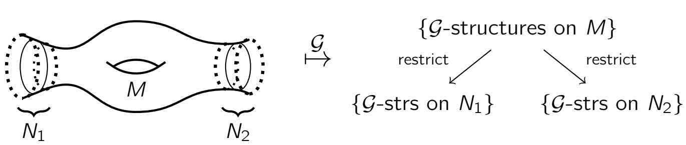
Idea of a defn: Fix $d \in \mathbb{N}$. Given "local structure" $\mathcal{G}$ (orientations, background fields, metrics, ...; probably $\mathcal{G}$ is a stack on site of d-manifolds and embeddings), imagine an (∞, d)-category $\mathrm{BORD}_d^{\mathcal{G}}$ with $\{k\text{-morphisms}\} = \{k\text{-manifolds with corners inside germs of } \mathcal{G}\text{-geometric } d\text{-manifolds}\}$. Imagine also an (∞, d)-category $d\mathrm{VECT}$ with $\{k\text{-morphisms}\} = \{\text{linear } (d-k-1)\text{-categories}\}$. A *$\mathcal{G}$-qft* is a symmetric monoidal functor $\mathrm{BORD}_d^{\mathcal{G}} \to d\mathrm{VECT}$.

Problems with defn: We don't quite have technology to fully define $d\mathrm{VECT}$ (we're close). For most $\mathcal{G}$, cannot yet define $\mathrm{BORD}_d^{\mathcal{G}}$ (units are an issue).

2. Definition of $\mathrm{BORD}_d^{\mathcal{G}}$ for sufficiently topological $\mathcal{G}$

Let $\mathrm{BORD}_d = \mathrm{BORD}_d^{\mathrm{smooth}}$. Lurie has explained how to define BORD_d. (C.f. Calaque–Scheimbauer.)

Given $\mathcal{G}$, can assign to each bordism in BORD_d a span (i.e. correspondence) of spaces:



When $\mathcal{G}$ is *sufficiently topological*, this assignment is a functor $\mathrm{BORD}_d \to \mathrm{SPANS}_d(\mathrm{SPACES})$ (= d-fold spans of spaces). ($\mathcal{G}$ a stack $\Rightarrow$ compositions. $\mathcal{G}$ is *suff. top.* if $\mathcal{G}$(def. retract) is equivalence. This is needed for units.)

Defn ($\mathrm{BORD}_d^{\mathcal{G}}$): Let $\mathrm{SPACES}_{\{\mathrm{pt}\}/} = \{\text{pointed spaces}\}$.

$$\begin{array}{ccc} \mathrm{BORD}_d^{\mathcal{G}} & \longrightarrow & \mathrm{SPANS}_d(\mathrm{SPACES}_{\{\mathrm{pt}\}/}) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{BORD}_d & \xrightarrow{\mathcal{G}} & \mathrm{SPANS}_d(\mathrm{SPACES}) \end{array}$$

Defn: $\mathrm{SPANS}_d(\mathrm{SPACES}; d\mathrm{VECT})$ is (∞, d)-category of "spans with local systems": a k-morphism is a k-fold span of spaces equipped with a flat bundle of k-morphisms in $d\mathrm{VECT}$. (C.f. Haugseng)

Lemma (Lurie): $\{\mathcal{G}\text{-qfts}\} := \{\mathrm{BORD}_d^{\mathcal{G}} \to d\mathrm{VECT}\} = \{\text{maps } \mathrm{BORD}_d \to \mathrm{SPANS}_d(\mathrm{SPACES}; d\mathrm{VECT}) \text{ covering } \mathrm{BORD}_d \xrightarrow{\mathcal{G}} \mathrm{SPANS}_d(\mathrm{SPACES})\}$.

Rmk: Such "qfts fibered over $\mathcal{G}$" are a version of "relative" (a.k.a. "twisted") field theories with "anomaly" (the linearization of) $\mathcal{G}$. I.e. linearize $\mathcal{G}$ to a functor $\mathcal{G} : \mathrm{BORD}_d \to (d+1)\mathrm{VECT}$; then qft fibered over $\mathcal{G}$ is nat. trans. $\mathbb{1} \Rightarrow \mathcal{G}$ where $\mathbb{1}$ is constant functor.

3. The cobordism hypothesis

Thm (Lurie):

$$\{\text{sym mon functors } \mathrm{BORD}_d \to \mathrm{SPANS}_d(\mathrm{SPACES})\} \simeq \{\text{spaces with homotopy } \mathrm{GL}(d, \mathbb{R})\text{-actions}\}$$

via $\mathcal{G} \mapsto \mathcal{G}(\{\mathrm{pt}\})$. The full *cobordism hypothesis* says:

$$\{\mathcal{G}\text{-qfts}\} \simeq \{\mathrm{GL}(d, \mathbb{R})\text{-equiv bundles over } \mathcal{G}(\{\mathrm{pt}\}) \text{ of "finite dimensional" linear } (d-1)\text{-cats}\}.$$

(Implicit assertion: $\mathrm{GL}(d, \mathbb{R})$ acts on $d\mathrm{VECT}^{\mathrm{fd}}$.)

Rmk: Usually stated for compact gp $\mathrm{O}(d) \overset{\sim}{\hookrightarrow} \mathrm{GL}(d, \mathbb{R})$, but more naturally about $\mathrm{GL}(d, \mathbb{R})$.

4. Main examples

E.g. (tangential structure): Given $G \to \mathrm{GL}(d, \mathbb{R})$, get coset space $\mathcal{G}(\{\mathrm{pt}\}) = \mathrm{GL}(d, \mathbb{R})/G$ with left $\mathrm{GL}(d, \mathbb{R})$-action. Corresponds to $\mathcal{G} : M \mapsto \{\text{prin. } G\text{-bundles } P \to M \text{ with } \mathrm{T}_M \cong P \times_G \mathbb{R}^d\}$.

Special cases:

- $G = \ker(\mathrm{sign} \circ \det : \mathrm{GL}(d, \mathbb{R}) \to \mathbb{Z}/2) \simeq \mathrm{SO}(d) \Rightarrow \mathcal{G}(\{\mathrm{pt}\}) = \mathbb{Z}/2$ and $\mathcal{G} = \{\text{orientations}\}$.
- G = double cover of above $\simeq \mathrm{Spin}(d) \Rightarrow \mathcal{G}(\{\mathrm{pt}\}) = \mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$ and $\mathcal{G} = \{\text{spin structures}\}$

These are the 0th and 1st entries in an infinite sequence:

Defn: $\mathrm{GL}(\infty, \mathbb{R}) = \bigcup_{d \to \infty} \mathrm{GL}(d, \mathbb{R})$ along $X \mapsto \left(\begin{smallmatrix} X & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Thm (Bott): $\mathrm{GL}(\infty, \mathbb{R})$ is homotopy abelian (i.e. infinite loop space) with all fundamental groups known. In particular $\pi_{\leq 0}\mathrm{GL}(\infty, \mathbb{R}) \simeq \mathbb{Z}/2$ and $\pi_{\leq 1}\mathrm{GL}(\infty, \mathbb{R}) \simeq \mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$. ($\pi_{\leq n}$ means the fundamental n-groupoid.)

Cor: $\{\text{orientations}\}$ and $\{\text{spin structures}\} : \mathrm{BORD}_d \to \mathrm{SPANS}_d(\mathrm{SPACES})$ correspond (via Cobordism Hypothesis) to left-multiplication action pulled back along $\mathrm{GL}(d, \mathbb{R}) \to \mathrm{GL}(\infty, \mathbb{R}) \to \pi_{\leq n}\mathrm{GL}(\infty, \mathbb{R})$ for $n = 0, 1$.

Rmk: As a G-space, left-multiplication action $G \curvearrowright G$ is the *trivial G-torsor*. In SPACES, all torsors are trivial.

5. Unitary qft

Other topoi allow nontrivial torsors. E.g. for field $\mathbb{F}$ with absolute Galois group $\mathrm{Gal}(\mathbb{F})$, set $\mathcal{X}$ = topos of stacks on site of commutative $\mathbb{F}$-algebras. Then $\{G\text{-torsors in } \mathcal{X}\} \simeq \mathrm{maps}(\mathrm{B}\,\mathrm{Gal}(\mathbb{F}), \mathrm{B}G)$.

Main e.g.: Since $\mathrm{Gal}(\mathbb{R}) = \mathbb{Z}/2$, there is a canonical nontrivial $\mathbb{Z}/2$-torsor over $\mathbb{R}$. It is $\mathrm{Spec}(\mathbb{C})$ with $\mathbb{Z}/2$-action by complex conjugation. Why? Because $\mathbb{C}$ = alg. closure of $\mathbb{R}$, and c.c. = $\mathrm{Gal}(\mathbb{R})$-action.

Connection to BORD_d: It makes perfect sense to talk about $\mathrm{SPANS}_d(\mathcal{X})$, etc., for any topos $\mathcal{X}$. $\mathrm{Spec}(\mathbb{C})$ with $\mathrm{GL}(d,\mathbb{R})$ action "c.c.∘(sign det)" defines, via Cobordism Hypothesis, a sym mon functor

$$\{\text{unitary}\} : \mathrm{BORD}_d \to \mathrm{SPANS}_d(\text{stacks over } \mathbb{R}\text{-algebras})$$

RHS is a category internal to $\mathcal{X}$ = stacks over $\mathbb{R}$-algebras, so pullback $\mathrm{BORD}_d^{\text{unitary}}$ is an internal category in $\mathcal{X}$. This means, e.g., that hom-spaces are $\mathbb{R}$-stacks.

Defn: *Unitary* qft is sym mon functor $Z : \mathrm{BORD}_d^{\text{unitary}} \to d\mathrm{VECT}_{\mathbb{R}}$, where $d\mathrm{VECT}_{\mathbb{R}}$ is the *internal* category in $\mathcal{X}$ of $\mathbb{R}$-linear $(d-1)$-categories.

If M unorientable, $\{\text{unitary}\}(M) = \emptyset$, since is $\mathbb{C}$-equivalent to $\{\text{orientations}\}(M)$. At each orientation of M, $Z(M)$ is bundle over $\mathrm{Spec}(\mathbb{C})$ — i.e. a $\mathbb{C}$-linear object — s.t. orientation reversal acts by complex conjugation.

Moreover, $Z(M \times$ $)$ is nondegenerate *sesquilinear* form on $Z(M)$. (Not necessarily positive definite.)

6. Spin–statistics

{Stacks over the site of com $\mathbb{R}$-algebras} is not the only topos of "stacks over $\mathbb{R}$." Consider instead "categorified commutative $\mathbb{R}$-algebras," i.e. $\mathbb{R}$-linear sym mon categories. Let $\mathcal{X}$ = stacks thereon.

Rmk: $\mathbb{R}$-algebra R categorifies to $(\mathrm{MOD}_R, \otimes_R)$.

Thm (Deligne): Among categorified com $\mathbb{R}$-algebras, $\mathbb{C}$ (i.e. $\mathrm{VECT}_{\mathbb{C}}$) is not algebraically closed. Rather, "categorified algebraic closure" of $\mathbb{R}$ is $\mathrm{SUPERVECT}_{\mathbb{C}}$.

Thm: Extension $\mathrm{VECT}_{\mathbb{R}} \to \mathrm{SUPERVECT}_{\mathbb{C}}$ is Galois with *categorified Galois group* $\mathrm{GAL}(\mathbb{R}) = \mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$.

$\mathbb{Z}/2$ acts by complex conjugation. $\mathrm{B}(\mathbb{Z}/2)$ acts by "$(-1)^f$," the endomorphism of identity functor that is $+1$ on even part and -1 on odd part (f = "fermion number").

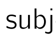
Thm: $\{G\text{-torsors in this } \mathcal{X}\} \simeq \mathrm{maps}(\mathrm{BGAL}(\mathbb{R}), \mathrm{B}G)$.

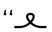
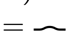
Rmk: For G a discrete group, $\mathrm{maps}(\mathrm{BGAL}(\mathbb{R}), \mathrm{B}G) = \mathrm{maps}(\mathrm{B}\,\mathrm{Gal}(\mathbb{R}), \mathrm{B}G)$, since $\pi_{\leq 0}\mathrm{GAL}(\mathbb{R}) = \mathrm{Gal}(\mathbb{R})$, so no new torsors. But for G a group among groupoids. . . .

Cor: In this $\mathcal{X}$, there is a canonical nontrivial $\mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$-torsor, namely $\mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{C}})$.

Defn: A *spin–statistics qft* Z is a $\mathcal{G}$-qft for $\mathcal{G}$ the functor $\mathrm{BORD}_d \to \mathrm{SPANS}_d(\mathcal{X})$ corr. to $\mathrm{Spec}(\mathrm{SUPERVECT}_{\mathbb{C}}) \in \mathcal{X}$ with $\mathrm{GL}(d,\mathbb{R})$-action via $\mathrm{GL}(d,\mathbb{R}) \to \mathbb{Z}/2 \ltimes \mathrm{B}(\mathbb{Z}/2)$.

(Implicit: $d\mathrm{VECT}_{\mathbb{R}}$ makes sense internal to this $\mathcal{X}$.)

Unpacking: If M is unspinnable, $\mathcal{G}(M) = \emptyset$, since it becomes equivalent to $\{\text{spin structures}\}(M)$ when base-changed to $\mathrm{SUPERVECT}_{\mathbb{C}}$. For M with spin structure, $Z(M)$ is a $\mathbb{C}$-linear *super* object (super vector space, super category, etc.), subject to rule $Z(M \times$ $) = (-1)^f|_{Z(M)}$.

Defn: " acts by $(-1)^f$" is the *spin–statistics theorem*. It says that *spinors* ((-1)-eigenstates of ) are *fermions* ((-1)-eigenstates of $(-1)^f$). N.b.:  = .

7. Even higher-categorical predictions

There is no reason to stop at "categorified com $\mathbb{R}$-algebras" i.e. sym mon *1*-categories over $\mathbb{R}$. For every n, there should be "n-categorified com $\mathbb{R}$-algebras" and an "n-categorified Galois group" $\mathrm{Gal}^{(n)}(\mathbb{R})$.

Expectation: $\mathrm{Gal}^{(n)}(\mathbb{R})$ is a group object among homotopy n-types, and projections $\mathrm{Gal}^{(n)}(\mathbb{R}) \to \mathrm{Gal}^{(n-1)}(\mathbb{R})$ comprise a Postnikov tower.

Defn: $\varprojlim \mathrm{B}\,\mathrm{Gal}^{(n)}(\mathbb{R})$ is the *∞-categorified étale homotopy type of* $\mathrm{Spec}(\mathbb{R})$.

Question: What is it? All I know is $\pi_1 = \pi_2 = \mathbb{Z}/2$.

Option: Perhaps $\prod_n \mathrm{B}^n(\mathbb{Z}/2)$?

That would be boring.

Option: Perhaps $\mathrm{BGL}(\infty,\mathbb{R})$?

If so, then completely explains unitarity, spin–statistics.

Option: Perhaps the connected component of the stable sphere, a.k.a. $\mathrm{BGL}(\infty,\mathbb{F}_1)$?

If so, then unitarity, spin–statistics are manifestations of the J-homomorphism.