Nonperturbative integrals, imaginary critical points, and homological perturbation theory

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To begin, I'd like to thank you all both for the invitation to speak, and also the hospitality this week. It's been a nice (if much too short) visit.

Details about what I'll tell you about are available at `arXiv:1206.5319`.

Introduction

A lot of the work in QFT involves making sense of "path integrals," which are integrals of the shape $\int_{\mathcal{M}} f\, e^s\, \mathrm{dVol}$, where $\mathcal{M}$ is some "space" of "fields," f is an "observable," s is an "action," and dVol is some "volume form" on $\mathcal{M}$. The problem is that the types of spaces in QFT tend not to admit analytic definitions of integration — they are infinite-dimensional, stacky, etc. The usual approach is either: (i) formal manipulation of the integral to predict results that make sense without an integral, or (ii) formal semiclassical expansion in Feynman diagrams.

What I'd like to do in this talk is to illustrate some techniques, which you might have seen in other contexts, that may be able to get us beyond those approaches. I'll focus on the simplest case. I'll give you one of the punchlines right now, which is that we'll end up writing down a polynomial / algebrogeometric version of the Batalin–Vilkovisky complex, although I won't call it that, and I won't derive it in the usual way (or maybe it is the usual way — I'm not really sure what the "usual" derivation of Batalin–Vilkovisky complex is).

Certainly BV complex is usually thought of as part of the perturbative/semiclassical theory. And in this talk there will be no power series.

Instead, I'll focus on the case when everything is a complex polynomial in n variables:

$$f, s \in \mathbb{C}[x_1, \dots, x_n]$$

What we're imagining is that s is fixed (it controls the physics) and f is varying (it's what we're measuring). Maybe we do or don't know s. Anyway, I'll assume that $\deg(s) = d$, and I will make a "genericity" assumption on s:

Assumption: Write $s = s^{(d)} + s^{(<d)}$, where $\deg(s^{(<d)}) \leq d-1$. So $s^{(d)}$ is the *top* part of s. I will assume that $s^{(d)}$ is *nonsingular*, in the sense that the hypersurface it defines in $\mathbb{P}^{n-1}$ is smooth. Equivalently, $s^{(d)}$ has non-vanishing discriminant.

It is, of course, very interesting to study what happens when you relax this assumption.

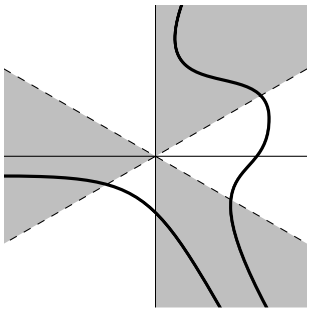
Anyway, I will take $\mathrm{dVol} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ the standard holomorphic n-form on $\mathbb{C}^n$. To define the integral $\int f\, e^s\, \mathrm{dVol}$ now requires choosing a *contour*

$$\gamma \hookrightarrow \mathbb{C}^n, \text{ an immersed } n\text{-real-dimensional submanifold}$$

and so that the integral converges, we should insist that e^s enjoys exponential decay at the ends of γ. The correct way to say this is that

$$\forall r \in \mathbb{R}\ \exists C \subseteq \gamma \text{ compact s.t. } \Re(s) < r \text{ on } \gamma \smallsetminus C$$

For example, when $n = 1$ and $s(x) = x^3$, here are some allowable contours:



Thus we can consider

$$\int_\gamma f\, e^s\, \mathrm{dVol}$$

Anyway, I don't know whether to call it **Cauchy's Theorem** of **Stokes' Theorem**, but the integral only depends on the homotopy type of γ. More precisely, it depends on the class of γ in the relative homology group

$$\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) \ll 0\})$$

I know two proofs, neither of which is due to me, of the following fact: assuming the genericity assumption above,

$$\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) \ll 0\}) \text{ is a free abelian group with dimension } (d-1)^n.$$

The algebra side

So s defines a pairing

$$\langle - \rangle_{s,\cdot} : \mathrm{H}_n(\mathbb{C}^n, \{\Re(s) \ll 0\}) \otimes \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$$

The left-hand factor is finite-dimensional and the right-hand factor is infinite-dimensional, so clearly there is a large kernel. We can recognize a chunk of the kernel (hopefully all):

Stokes' Theorem: $\int f\, e^s = 0$ if $f = \frac{\partial g}{\partial x_i} + \frac{\partial s}{\partial x_i} g$ for some $g \in \mathbb{C}[x_1, \dots, x_n]$ and $i = 1, \dots, n$. I.e. the pairing $\langle - \rangle_{s,\cdot}$ factors in the right-hand spot through:

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{\sum_i \operatorname{im}\big(\partial_i + \partial_i s \cdot\big)} \quad \text{(the "Stokes quotient")}$$

Important to emphasize: the sum is not direct, and this is not an algebra quotient — the denominator is not an ideal — just a vector-space quotient.

Problem-Solving Technique: To understand subquotients, find them as homology groups of naturally occurring chain complexes.

We set $V_0 = \mathbb{C}[x_1, \dots, x_n]$. Then the description above basically tells us V_1:

$$V_1 = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \dots, x_n], \qquad \partial = \bigoplus_i \big(\partial_i + \partial_i s \cdot\big)$$

But now we have lots of homology in degree 1. The correct complex to use is like a Koszul complex for an intersection:

$$\begin{aligned} V_\bullet &= \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n], \text{ where the } \xi_i\text{s have homological degree 1, hence anticommute} \\ &= \text{polynomial antisymmetric multivectorfields on } \mathbb{C}^n \text{ with wedge multiplication} \end{aligned}$$

$$\partial_{\text{full}} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i} \right) \frac{\partial}{\partial \xi_i}$$

I'll call it "∂_{full}" and not just "∂" because there will be lots of different differentials.

Important note: $V_\bullet$ is a graded commutative algebra, but $(V_\bullet, \partial)$ *is not a dga*, because ∂ is not a derivation (it's a second-order differential operator).

Problem-Solving Technique (homological perturbation lemma): So now here's the most important problem-solving technique in the talk, so if you've been snoozing, wake up, and then go back to sleep afterwards. The idea is that to understand a differential, break it up into a simpler part plus a perturbation. Some definitions:

Definition:

- A *retraction*, for the purposes of this talk, consists of chain complexes which I will conveniently call $(H_\bullet, \partial_H)$ and $(V_\bullet, \partial_V)$, and chain maps $\varphi : H \to V$ and $\tau : V \to H$, such that $\tau\varphi = \mathrm{id}_H$, and although $\varphi\tau$ is not the identity on V, it is homotopic to the identity with homotopy parameterized by η: $\varphi\tau = \mathrm{id}_V - [\partial_V, \eta]$.

$$(H_\bullet, \partial_H) \underset{\varphi}{\overset{\tau}{\leftrightarrows}} (V_\bullet, \partial) \circlearrowright \eta \qquad \begin{aligned} \tau\varphi &= \mathrm{id}_H \\ \varphi\tau &= \mathrm{id}_V - [\partial, \eta] \end{aligned}$$

(I won't demand any "side conditions" or anything.)

- A *perturbation* of a chain complex $(V_\bullet, \partial_V)$ is a *Maurer–Cartan element of* $\mathrm{End}(V)$, i.e. it is $\delta \in \mathrm{End}(V)$ such that $\partial + \delta$ is a differential on $V_\bullet$, i.e. δ is of homological degree $+1$ and $[\partial, \delta] + \frac{1}{2}[\delta, \delta] = 0$.
- A perturbation δ is *small* with respect to a retraction $H \leftrightarrows V \circlearrowleft$ if $(\mathrm{id} - \delta\eta)$ is invertible.

Homological perturbation lemma: The HPL is certainly not due to me — it's from the 1960s, and the best write-up I know is Crainic's 2004 paper. It's basically a version of spectral sequences that gives you more explicit and algorithmic control. It says that given a retraction as above and a small perturbation, you get a new retraction:

$$\left(H_\bullet, \tilde\partial = \partial_H + \tau(\mathrm{id} - \delta\eta)^{-1}\delta\varphi\right) \underset{\tilde\varphi = \varphi + \eta(\mathrm{id}-\delta\eta)^{-1}\delta\varphi}{\overset{\tilde\tau = \tau(\mathrm{id}-\delta\eta)^{-1}}{\leftrightarrows}} (V_\bullet, \partial + \delta) \circlearrowleft \tilde\eta = \eta(\mathrm{id}-\delta\eta)^{-1}$$

Proof: check a bunch of equations.

So, in our application, I'll take $V_\bullet = \mathbb{C}[x_1, \dots, \xi_n]$ as above. I want $\partial + \delta = \partial_{\mathrm{full}}$. Then I want ∂ to be something simple, and δ to be small.

Here's what I'll choose. Recall the top part $s^{(d)}$. Then I'll take:

$$\partial = \partial_{(d)} = \sum_i \frac{\partial s^{(d)}}{\partial x_i} \frac{\partial}{\partial \xi_i}, \qquad \delta = \sum_i \left(\frac{\partial}{\partial x_i} + \frac{\partial s^{(<d)}}{\partial x_i} \right) \frac{\partial}{\partial \xi_i}$$

The reason is because this ∂ makes $(V_\bullet, \partial_{(d)})$ into precisely the *Koszul complex* for the intersection

$$\bigcap_{i=1}^n \mathrm{spec}\big(\mathbb{C}[x]/(\partial_i s^{(d)})\big) = \{\mathrm{d}s^{(d)} = 0\}$$

the *scheme-theoretic critical locus* of $s^{(d)}$. Now, because $s^{(d)}$ is nonsingular, this is in fact a complete intersection. Then **Bezout's Theorem** says that $\mathrm{H}_0(V_\bullet, \partial_{(d)}) = \mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$ is $(d-1)^n$-dimensional, and **Serre's Theorem** says that all other homology groups vanish. (I could have used s instead....)

Anyway, $H_\bullet = \mathrm{H}_\bullet(V_\bullet, \partial_{(d)})$. Then τ is 0 except in homological degree 0, where we take it to be the *restriction* map $\mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$.

Now I need to choose φ and η so that δ is small. Here's how. Given $V_\bullet$ an addition $\mathbb{N}$-grading by setting $\deg(x_i) = 1$ and $\deg(\xi_i) = d - 1$. Then $\partial_{(d)}$ preserves the $\mathbb{N}$-grading. So we can arbitrarily choose φ to split τ as a map of *graded* vector spaces. And we can always choose the homotopy η to preserve the $\mathbb{N}$-grading.

Ok, but notice that δ strictly lowers the $\mathbb{N}$-grading, so $\delta\eta$ acts locally nilpotently on $V_\bullet$, so $(\mathrm{id} - \delta\eta)$ is invertible. Thus we can apply HPL.

Remark: Ah, but $H_\bullet$ is concentrated in degree 0. So it doesn't have room to deform — it's "discrete" in the homological sense. Moreover,

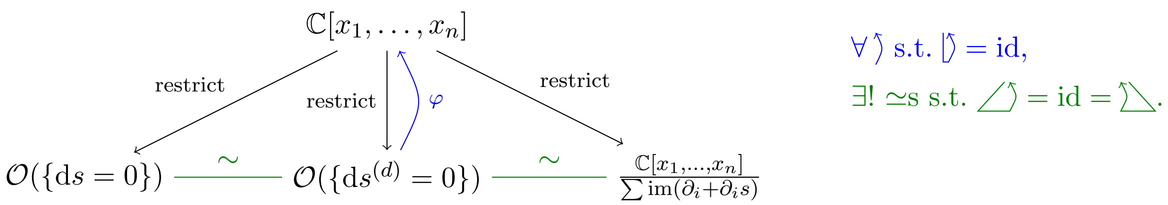
$$\tilde{\varphi} = \varphi + (\dots)\delta\varphi = \varphi,$$

because φ only lives in homological degree 0 and δ lowers homological degree. Moreover,

$\tilde{\tau}$ is determined by the equation $\tilde{\tau} \circ \varphi = \mathrm{id}_H$
and the condition that it be a chain map $(V_\bullet, \partial_{\mathrm{full}}) \to H_\bullet$

We could also have used $\delta = \partial_{(<d)} = \sum_i \frac{\partial s^{(<d)}}{\partial x_i} \frac{\partial}{\partial \xi_i}$, whence $\partial_{(d)} + \partial_{(<d)}$ is the differential for the Koszul complex for $\{\mathrm{d}s = 0\}$. All together, this proves:

Theorem: For each polynomial-degree-preserving splitting φ of the restriction map $\mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$, there exist unique isomorphisms $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\}) \cong \mathcal{O}(\{\mathrm{d}s^{(d)} = 0\}) \cong \frac{\mathbb{C}[x_1,\dots,x_n]}{\sum \mathrm{im}(\partial_i + \partial_i s)}$ such that φ also splits the restriction maps to $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$ and to $\frac{\mathbb{C}[x_1,\dots,x_n]}{\sum \mathrm{im}(\partial_i + \partial_i s)}$. (All splittings are as vector space maps only — generally algebra homomorphisms do not exist, and $\frac{\mathbb{C}[x_1,\dots,x_n]}{\sum \mathrm{im}(\partial_i + \partial_i s)}$ isn't even an algebra.)



I wish I could remove the "grading-preserving" condition.

Interpretation: The choice of splitting φ is analogous to choosing a way to fiber $\mathbb{C}^n$ over $\{\mathrm{d}s = 0\}$. Of course, this cannot be done as schemes — that's what φ is not an algebra homomorphism. Then the deformed map $\tilde{\tau}$ is the map that "integrates out the fibers," giving an "effective action" on the classical vacua. Then the contour γ picks out a measure on the classical vacua, but only after you've chosen the fibration, which is as it is in the usual situation.

Concluding remarks

I'm sure I've already run out of time, so let me just mention a few more things.

Theorem: Using the same techniques, but under a much stronger genericity condition on $s^{(d)}$, you can uniformly (and highly non-canonically) choose the splitting φ. Namely, the three vector spaces above each have a basis consisting of the restrictions of the monomials $x_1^{m_1} \cdots x_n^{m_n}$ which are deparately of degree $< d-1$ in each variable, i.e. $m_i < d-1\ \forall i$. Then you can define φ on this basis to be $\varphi([x_1^{m_1} \cdots x_n^{m_n}]) = x_1^{m_1} \cdots x_n^{m_n}$, and the isomorphisms are the identity for this basis.

Remarks on Chern–Simons theory: I mentioned the Volume Conjecture in the abstract, so let me say a few words. Certainly I have no results, just some directions to look. The Volume Conjecture predicts that certain limits of the Chern–Simons path integral are dominated by imaginary critical points — the Jones polynomial should be computed by an integral of the form $\int f\, e^s$ where s is the Chern–Simons functional, and we usually think of it as an integral over the space of SU(2) connections, but the dominant contribution in the limit in Volume Conjecture is from a flat $\mathrm{SL}(2,\mathbb{R})$ connection. A year or two ago Witten wrote a hundred-page paper exploring this and related mysteries.

However, the moral of the above work is that this isn't at all mysterious. s is a polynomial, and if f is polynomial (or polynomial times exp(quadratic)), then it should not be a surprise that all the scheme-theoretic critical points contribute. This is in stark contrast to the semiclassical/perturbative case, where the same techniques give you Feynman diagrams but the role of $\{\mathrm{d}s = 0\}$ is played by the smooth (real) critical locus. Now, Witten's paper has lots of other things in it — he considers $\mathrm{SL}(2,\mathbb{C})$ both as an algebraic object and as a smooth object. See, SU(2) and $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ are almost indistinguishable from the point of view of polynomial functions, but to smooth functions they look very different. So you should think about what types of functions you actually care about.

What I'd like to be able to do is use similar techniques to write down a homological definition of the non-perturbative Chern–Simons path integral. And perhaps verify that it satisfies the Jones Polynomial skein relation. What you'd have to do is to choose (you're in infinite dimensions, so there are lots of choices) some algebrogeometric version of the space of connections, and there are derived-algebraic-geometry ways of taking the quotient modulo gauge. Then you should work out what is the Koszul complex for the subspace of flat connections.

This can all be done, in too many ways. The hard part is to find any perturbation at all that deserves to be thought of as "doing the integral" — the $\sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}$ part of ∂_{full}. Presumably, if you have one, then the same game with $\mathbb{N}$-gradings will make it small. What gets in the way of just writing it down naively is the same thing that in the case of perturbative integrals causes *ultraviolet divergences*. For perturbative integrals, there is a well-understood way of trying to fix these, by working order-by-order in Planck's constant, but that's not available here. Now, maybe it can be done — in fact Chern–Simons theory doesn't really have UV divergences. I haven't tried too hard yet, but it's one of the directions to look.