Phases of 2d sqfts

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all jt w/ D. Gaiotto, and ^ jt w/ E. Witten.

Plan

1. Warm-up: Anomalies, QM, and K-theory
2. Small-index sqfts
3. Mock modularity.

0. A question many physicists have asked:

$\boxed{\text{classify, up to deformation, quantum systems of some type.}}$

If you care about deformation classes, you are really asking:

$\boxed{\text{what is the homotopy type of the moduli space of quantum systems?}}$

Of course, the answer depends on what types of QFTs you like:
- dimension?
- # susys?
- gapped / not gapped?

For me, I ultimately care about

$$\left\{ (1+1)d \quad N=(0,1) \text{ ``vegnize'' } SQFTs \right\}.$$

I don't demand any other symmetry (eg. conformal).

"very nice" is some as-yet-to-be-determined analytic conditions: unitary, "compact", etc.
(so can wick-rotate)
≈ discrete spectrum

<u>Remark about deformations</u> (cf. <u>Seiberg's talk at Strings 2019</u>):
Traditional $\boxed{\text{hep-th}}$ notion of "deformation" is
* add a marginal or relevant operator,
* RG flow a finite amount.

I ~~also~~ want more of the $\boxed{\text{cond-mat}}$ version.
In particular, I ask that a QFT be deformation-equivalent to its far-IR limit. So for me allowed ~~to~~ deformations include
* tensor your QFT w/ any QFT in the trivial gapped phase.

1. Warm-up

Start w/ bosonic (0+1)d QFTs, ie QM models.
What is their moduli space? Two versions of the question:

~~Schrodinger~~

$$\left\{\begin{matrix}\text{nonanomalous} \\ \text{nice QM models}\end{matrix}\right\} \qquad\qquad \left\{\begin{matrix}\text{mildly anomalous} \\ \text{nice QM models}\end{matrix}\right\}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\left\{\begin{matrix}\text{sep. Hilbert space } \mathcal{H} \\ \text{w/ self-adjoint operator } \hat{H} \\ \text{s.t. } \exp(-t\hat{H}) \text{ is} \\ \text{traceclass } \forall t>0.\end{matrix}\right\} \qquad \left\{\begin{matrix}\text{Type-I factor } \mathcal{A} \\ \text{w/ self-adjoint } \hat{H} \in \mathcal{A} \\ \text{s.t. } \exp(-t\hat{H}) \text{ is} \\ \text{traceclass.}\end{matrix}\right\}$$

$$\text{is} \qquad\qquad\qquad\qquad \text{is}$$

$$* \; BU(\infty) \simeq * \qquad\qquad BPU(\infty) \simeq K(\mathbb{Z}, 3).$$

"Schrodinger picture"   "Heisenberg picture".

Eventually I want some susy. First let's add fermions:

$\left\{ \text{nonanom fermionic QMs} \right\}$ = $\left\{ \text{super hilbert space + ham.} \right\}$ IS $*$

$\left\{ \text{mildly anomalous fermionic QMs} \right\}$ = $\left\{ \text{super type-I factor + ham.} \right\}$ IS $K(\mathbb{Z},3) \cdot K(\mathbb{Z}_2,1) \cdot K(\mathbb{Z}_2,0)$

"$(-1)^f$".

up to Bo, two "Type I" factors, $\sim \mathbb{C}$, Cliff(1).

this is where anomalies of (0+1)d spin theories live.

$\boxed{\text{N.B.: nontrivial extensions.}}$

Moral: In the absence of any susy, the anomaly is the only def invariant.

Now turn on minimal susy:

$\left\{ \text{nonanom } N=1 \text{ SQMs} \right\}$ = $\left\{ \mathcal{H} \text{ plus an odd self-adjoint op. } \hat{G} \text{ s.t. } \exp(-t\hat{G}^2) \text{ is trace class} \right\}$

Not contractible, e.g. has an index $\to \mathbb{Z}$.

In fact: IS KU_0.

Can get all of KU_0 by allowing anomalies.

By analogy:

non anomalous fermionic (1+1)d QFTs — **Conjecture:** is $*$

mildly anomalous fermionic (1+1)d QFTs — **Conjecture:** is $K(\mathbb{Z},4) \cdot K(\mathbb{Z},2) \cdot K(\mathbb{Z},1) \cdot K(\mathbb{Z},0)$

"grav. anomaly" $n = 2(c_L - c_R)$.

E.g.: $\mathrm{Fer}(n)$ = n leftmoving ~~stra~~ majorans has $2(c_L - c_R) = n$.

The conjectures assert that the only def-invariant of very nice non-susy QFTs is the anomaly.

~~Defn: $SQFT_n = \mathbb{Z}$~~

Whereas if we turn on minimal aka $N=(0,1)$ susy, then definitely not contractible.

2. Small-index SCFTs

<u>Defn</u>: $SQFT_n = \left\{ \begin{array}{l} N=(0,1) \ (1+1)d \ SQFTs \\ \text{w/ anomaly} \equiv \text{anomaly of } \mathrm{Fer}(n) \end{array} \right\}$

↑ data: must choose an equiv.

First invariant:

The <u>index</u>, aka <u>elliptic genus</u>, is the partition function on Ramond-Ramond tori.

Standard argument (requires "compactness"):

$$Z_{RR} : SQFT_n \longrightarrow MF_{n/2}$$

is a def. invariant.

weakly holomorphic modular forms of wt $n/2$.

N.B: 0 unless $8|n$.

my fav. example: Purely antiholo SQFT $\cong$ $N=1$ holo SVOA

$$\mapsto \underset{\overset{\|}{\Delta^{n/24}}}{\eta^n} \cdot (\#\ \text{R-sector ground states})$$

Standard example: [Moore: required to define the σ-model]

if M is a riemannian manifold w/ string str, then

$$\underset{\text{w/ target } M}{(0,1)\ \sigma\text{-model}} \longrightarrow \mathrm{Wit}(M).$$

"Stolz–Teichner" hypothesis":

have

$$\begin{tikzcd} & SQFT_\bullet \arrow[dd, "\text{is}"'] \arrow[dr] & \\ MString_\bullet \arrow[ur] \arrow[dr] & & MF_{\bullet/2} \\ & TMF_\bullet \arrow[ur] & \end{tikzcd}$$

↑ universal elliptic cohomology

Our papers provide evidence for this hypothesis.
~~E.g.~~ Specifically: The hypothesis leads to "numerological" predictions about $SQFT_\bullet$. Then we try to verify them.

For example, $\pi_n TMF \to MF_{n/2}$ is not surjective. E.g., $\Delta \notin$ image. In fact, $m\Delta^d \in$ image iff $24 \mid md$.

Conjecture: $\forall d$, $\exists$ antiholomorphic SCFT of grav anom $2c_R = 24d$ and index $m = 24/\gcd(24,d)$.

Example: "conway moonshine" has $d=1$, $m=24$.

Using computer, we found solns for $d \le 5$. We don't know a systematic construction.

3. Mock modularity

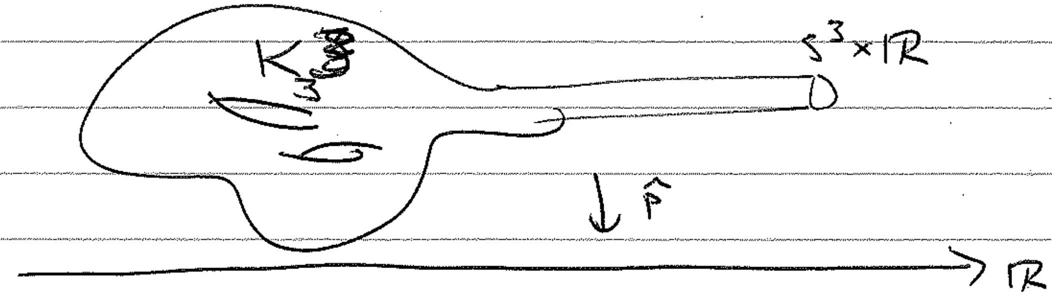
String strs (on a Spin manifold) are a torsor for H^3. For round $M^* = S^3$, $\exists$ unique reflection-inv String str, giving an origin to this $\mathbb{Z}$-torsor. Consider round S^3 w/ String str K. Expect:

- Far IR limit of the σ-model
  $= N=(0,1)$ s WZW model at bosonic level $(|K|-1)$
  $= \chi WZW(SU(2), K) \times \overline{\chi WZW(SU(2),K)} \times \overline{Fer(3)}$.

Reason for expectation: $*$ perturbation thy in $K \to \infty$
$*$ anomaly matching for $SO(4)$-action.

ST hyp $\Longrightarrow$ This model can be deformed to one w/ spont. susy breaking iff $24 | K$.

if direction: $(K3 - pt)$ has unique String str. It restricts to nbhd of pt to $K=24$.



Noncompact σ-model $\Longrightarrow$ not "very nice"!
Pick $p \in \mathbb{R}$. Consider "quantum ham reduction" to force $\hat{p} = p$. This is explicit: stick in a left-moving fermion λ, and add $\hat{p}\lambda$ to

the superpotential. This λ acts (in the IR) as a Lagrange multiplier to force $\hat{p} \equiv p$.

When $p \gg 0$, this theory is round S^3.
When $p \ll 0$, this thy has spont. susy breaking.

Remark: more generally:

$$\text{path in } SQFT_n \quad \underset{\longleftarrow}{\longrightarrow} \quad \text{"noncompact" } SQFT_{n-1}$$
$p \mapsto \mathcal{F}(p)$ w/ cylindrical ends
(const. near $\pm\infty$)

~~changes grav. anom~~

$\longleftarrow$: Lagrange multiplier ($\lambda \rightsquigarrow c_L$ goes up by $+1/2$)
$\longrightarrow$: "dynamicallyze p" (superpartner of p $\rightsquigarrow c_R$ goes up by $+1/2$)

These are a homotopy equiv. $\Rightarrow$ $SQFT_\bullet$ is an Ω spectrum.

What about the "only if" direction? Need an invariant of SQFTs w/ anomaly 3.

<u>Hint</u> (<u>Berwick-Evans, Bunke + Naumann</u>):

$$\begin{tikzcd}
SQFT_\bullet \arrow[dr, dotted] \arrow[rr, "Z_{RR}"] \arrow[ddr, bend right, "\text{compactify on } S^1"'] & & HMF_\bullet^{\mathbb{C}} \arrow[dd] \\
& \text{"KOMF"}_\bullet \arrow[ur] \arrow[d, "\Gamma"] & \\
& KO_\bullet((q)) \arrow[r] & H\mathbb{C}((q))_\bullet
\end{tikzcd}$$

de Rham coh w/ coeffs in $MF_\bullet$

S^1-equiv KO!

4-periodized

This is a homotopy pullback, and so

$$\pi_{4k-1}\,\mathrm{KOMF} = \frac{\mathbb{C}(\!(q)\!)}{\mathbb{Z}(\!(q)\!) + \mathrm{MF}^{\mathbb{C}}_{2k}} \neq 0$$

where $\mathbb{C}(\!(q)\!) = \pi_{4k} H\mathbb{C}(\!(q)\!)$, $\mathbb{Z}(\!(q)\!) = \pi_{4k} KO(\!(q)\!)$, $\mathrm{MF}^{\mathbb{C}}_{2k} = \pi_{4k} H\mathrm{MF}_{\circ}$

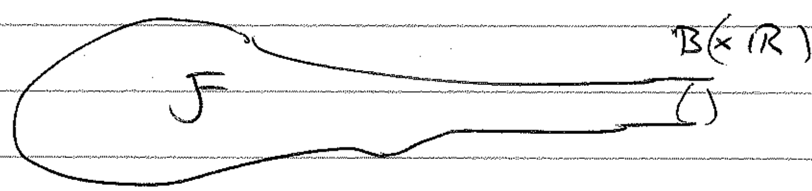
and so expect a "secondary invariant"

$$\pi_{4k-1}\,\mathrm{SQFT} \longrightarrow \frac{\mathbb{C}(\!(q)\!)}{\mathbb{Z}(\!(q)\!) + \mathrm{MF}_{2k}}$$

Davide and I proposed a SQFT description of this invariant.

$\cancel{\partial n} = \frac{3}{c_{24}}$

Consider a nullhomotopy of a "boundary" thy $\mathcal{B}$.
Package as a degree-4 noncompact SQFT $\mathcal{F}$ $\in \mathrm{SQFT}_3$
w/ cylindrical end:



Look at $\hat{f} = Z_{RR}(\mathcal{F})$. It is manifestly real-analytic-modular. Is it holomorphic? Usual argument.

$$\frac{\partial \hat{f}}{\partial \bar{z}} = \langle T_{\bar{z}\bar{z}} \rangle = \langle [\bar{G}_{\bar{z}}, \bar{G}_{\bar{z}}] \rangle$$

$C^{(0,1)}$ susy.

~~[illegible]~~, why does $\langle [x, y] \rangle = 0$?. Integrate by parts. If compact, ✓. In general, should pick up a boundary term. (boundary in field space.) ~~So~~ Claim:

$$\langle \bar{G}[\mathcal{O}] \rangle_{\mathcal{F}} \propto \langle \mathcal{O} \rangle_{\mathcal{B}}.$$

This ~~[illegible]~~ predicts a holomorphic anomaly (assuming $\mathcal{B}$ is a CFT, i.e. in the far IR).

$$\sqrt{-8\tau_2}\,\frac{\partial \hat{f}}{\partial \bar{\tau}} = (\eta^n) \langle \bar{G} \rangle_{\mathcal{B}}$$

we checked the coeff in a bunch of examples

Also in a compact thy, Z_{RR} would have (even) integral ~~[illegible]~~ q-expansion. Why? "Compactify on S^1". Breaks manifest modularity, but gives integrality. ~~Actually [illegible]~~ If you look more carefully, this "compactify on S^1 and take the index" corresponds to the limit

$$f(\tau) = \lim_{\bar{\tau} \to -i\infty} \hat{f}(\tau, \bar{\tau}).$$

Then f will be integral (modulo η-invariant-related correction.) ~~[illegible]~~

~~the correction~~ "mod-2 index of $\mathcal{B}$".

So we learn:

* Given B, compute $g(\tau,\bar\tau) = \langle \bar G \rangle_B$
(and compute a mod-2-index, which usually vanishes).

* If B is null homotopic, then $\exists\, \hat f$ s.t.
$\sqrt{-8\tau_2}\,\partial \hat f = g$. $\quad f = \lim_{\bar\tau \to -i\infty} \hat f \in \mathbb{Z}(q)$.

~~Conversely~~, ~~i.e.~~

N.B.: g determines $\hat f$ (~~and hence f~~)
~~up to~~ modulo MF,
so $g \mapsto [f] \in \frac{\mathbb{C}(q)}{MF}$.

So, if $[f] \neq 0 \in \frac{\mathbb{C}(q)}{MF + \mathbb{Z}(q)}$, then not null homotopic.

This class is the secondary invariant.

For ~~S^3~~ S^3_K, g is ~~an explicit~~ because of rationality of the CFT,
a dot product of two specific vector-valued
modular forms $g = g_L \cdot \bar g_R$,

and can explicitly solve $\hat f = g_L \cdot \hat f_R$
for $\hat f_R$ a specific vector-valued mock modular form
w/ ~~shadow g_R~~ shadow g_R,

and remarkably

$$f = g_L \cdot f_R = \frac{-K}{24} + \mathbb{Z}(q)\,!$$