Nonperturbative integrals, imaginary critical points, and homological perturbation theory

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Thank you for the invitation to speak. Details about what I'll tell you about are available at `arXiv:1206.5319`.

Introduction

A lot of the work in QFT involves making sense of "path integrals," which are integrals of the shape $\int f\, e^s$. Here s is an "action" that controls the physics, and f is an "observable" that we choose to measure. The usual approach is either: (i) formal manipulation of the integral to predict results that make sense without an integral, or (ii) formal semiclassical expansion in Feynman diagrams.

What I'd like to do in this talk is to illustrate some techniques, which you might have seen in other contexts, that may be able to get us beyond those approaches. I'll give you one of the punchlines right now, which is that we'll end up writing down a polynomial / algebrogeometric version of the Batalin–Vilkovisky complex. I'll focus on the simplest case:

Set-up: For this talk (the paper is more general), $f, s \in \mathbb{C}[x_1, \dots, x_n]$ are complex polynomials in n variables. I will assume that s is homogeneous of degree d. Moreover, I will assume the following *genericity assumption*, that the hypersurface $\{s = 0\} \subseteq \mathbb{CP}^{n-1}$ is non-singular, or equivalently that s has nonvanishing discriminant.

Then to define the integral also requires choosing a contour, which I don't have time to talk about. But I'll mention a result that I know at least two proofs of (neither due to me):

Fact: Assuming the genericity assumption, the pertinent space of contours is a free abelian group on $(d-1)^n$ generators. Remember that number.

Thus we have a pairing

$$\{\text{contours}\} \otimes \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$$

which of course must have kernel in the second spot. We can recognize part of the kernel:

Stokes' Theorem: $\int f\, e^s$ vanishes if $f\, e^s = \frac{\partial}{\partial x_i}\big(g\, e^s\big)$ for some $i = 1, \dots, n$ and $g \in \mathbb{C}[x_1, \dots, x_n]$. Put another way, the pairing factors in the second spot through the quotient vector space

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{\sum_{i=1}^n \operatorname{im}\big(\frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i}\big)}$$

Note that the denominator is not a direct sum, and is not an ideal.

Problem-Solving Technique: Scott already mentioned a very important problem-solving technique: To understand subquotients, find them as homology groups of naturally occurring chain complexes.

A good complex to use is like a Koszul complex for an intersection:

$$V_\bullet = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n] = \Gamma\big(\mathrm{T}^{\wedge\bullet}\mathbb{C}^n\big)$$

where $|\xi_i| = 1$ and $|x_i| = 0$.

$$\partial_{\text{full}} = \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i}\right) \frac{\partial}{\partial \xi_i}$$

I'll call it "∂_{full}" and not just "∂" because there will be lots of different differentials.

Important note: $V_\bullet$ is a graded commutative algebra, but $(V_\bullet, \partial)$ *is not a dga*, because ∂ is not a derivation (it's a second-order differential operator).

Why are chain complexes useful? One reason is that they are more robust than vector spaces under deformation. See, ∂_{full} is somewhat complicated. But there is another

Problem-Solving Technique: Hope that

$$\text{complicated} = \text{simple} + \text{small}$$

Let me make this precise, using something Niels already referred to, namely *homological perturbation theory*. This is the most important problem-solving technique in the talk, so if you've been snoozing, wake up, and then go back to sleep afterwards. Some definitions:

Definition:

- A *retraction*, for the purposes of this talk, consists of chain complexes which I will conveniently call $(H_\bullet, \partial_H)$ and $(V_\bullet, \partial_V)$, and chain maps $\varphi : H \to V$ and $\tau : V \to H$, such that $\tau\varphi = \mathrm{id}_H$, and although $\varphi\tau$ is not the identity on V, it is homotopic to the identity with homotopy parameterized by η: $\varphi\tau = \mathrm{id}_V - [\partial_V, \eta]$.

$$(H_\bullet, \partial_H) \underset{\varphi}{\overset{\tau}{\leftrightarrows}} (V_\bullet, \partial) \circlearrowleft \eta \qquad \begin{aligned} \tau\varphi &= \mathrm{id}_H \\ \varphi\tau &= \mathrm{id}_V - [\partial, \eta] \end{aligned}$$

  (I won't demand any "side conditions" or anything.)

- A *perturbation* of a chain complex $(V_\bullet, \partial_V)$ is $\delta \in \mathrm{End}(V)$ such that $\partial + \delta$ is a differential on $V_\bullet$.
- A perturbation δ is *small* with respect to a retraction $H \leftrightarrows V \circlearrowleft$ if $(\mathrm{id} - \delta\eta)$ is invertible.

Homological Perturbation Lemma: The HPL is certainly not due to me — it's from the 1960s, and the best write-up I know is Crainic's 2004 paper. It's basically a version of spectral sequences that gives you more explicit and algorithmic control. Alternately, it's the homotopy perturbation theory for the algebraic structure of "a choice of Maurer–Cartan element." It says that given a retraction as above and a small perturbation, you get a new retraction:

$$\big(H_\bullet, \tilde\partial = \partial_H + \tau(\mathrm{id} - \delta\eta)^{-1}\delta\varphi\big) \underset{\tilde\varphi = \varphi + \eta(\mathrm{id}-\delta\eta)^{-1}\delta\varphi}{\overset{\tilde\tau = \tau(\mathrm{id}-\delta\eta)^{-1}}{\leftrightarrows}} (V_\bullet, \partial + \delta) \circlearrowleft \tilde\eta = \eta(\mathrm{id}-\delta\eta)^{-1}$$

Proof: check a bunch of equations.

So, in our application, I'll take $V_\bullet = \mathbb{C}[x_1, \dots, \xi_n]$ as above. I want $\partial + \delta = \partial_{\text{full}}$. Then I want ∂ to be something simple, and δ to be small.

Application to the problem at hand: Here's what I'll choose. Recall the top part $s^{(d)}$. Then I'll take:

$$\partial = \partial_{\text{cl}} = \sum_i \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \xi_i}, \qquad \delta = \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}$$

The "cl" is for "classical," because $(V_\bullet, \partial_{\text{cl}})$ is precisely the *Koszul complex* for the intersection

$$\bigcap_{i=1}^n \mathrm{spec}\big(\mathbb{C}[x]/(\partial_i s)\big) = \{\mathrm{d}s = 0\}$$

the *scheme-theoretic critical locus* of s. Now, because s is nonsingular, this is in fact a complete intersection of n degree-$(d-1)$ hypersurfaces. Then **Bezout's Theorem** says that $\mathrm{H}_0(V_\bullet, \partial_{\text{cl}}) = \mathcal{O}(\{\mathrm{d}s = 0\})$ is $(d-1)^n$-dimensional, and **Serre's Theorem** says that all other homology groups vanish.

Anyway, $H_\bullet = \mathrm{H}_\bullet(V_\bullet, \partial_{\text{cl}})$. Then τ is 0 except in homological degree 0, where we take it to be the *restriction* map $\mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\{\mathrm{d}s = 0\})$.

Corollary: If φ, η can be chosen so that δ is small, then since $H_\bullet = H_0$ and $V_\bullet = V_{\geq 0}$, it follows that

$$\tilde\varphi = \varphi + (\dots)\delta\varphi = \varphi,$$

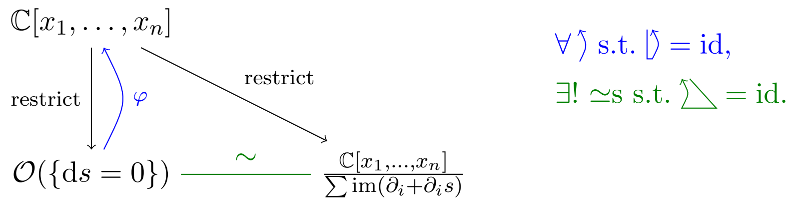
because δ lowers homological degree, and

$\tilde\tau$ is determined by the equation $\tilde\tau \circ \varphi = \mathrm{id}_H$
and the condition that it be a chain map $(V_\bullet, \partial_{\text{full}}) \to H_\bullet$

So the choice of η is important to know that it's small, and to write explicit formulas, but actually φ is the only real data.

Thus we can prove:

Theorem: For each polynomial-degree-preserving splitting φ of the restriction map $\mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\{\mathrm{d}s = 0\})$, there exist unique isomorphisms $\mathcal{O}(\{\mathrm{d}s = 0\}) \cong \frac{\mathbb{C}[x_1,\dots,x_n]}{\sum \mathrm{im}(\partial_i + \partial_i s)}$ such that φ also splits the restriction maps to $\frac{\mathbb{C}[x_1,\dots,x_n]}{\sum \mathrm{im}(\partial_i + \partial_i s)}$. (All splittings are as vector space maps only — generally algebra homomorphisms do not exist, and $\frac{\mathbb{C}[x_1,\dots,x_n]}{\sum \mathrm{im}(\partial_i + \partial_i s)}$ isn't even an algebra.)



I wish I could remove the "polynomial-degree-preserving" condition. I don't know how to assure smallness without it.

Interpretation: The choice of splitting φ is analogous to choosing a way to fiber $\mathbb{C}^n$ over $\{\mathrm{d}s = 0\}$. Of course, this cannot be done as schemes — that's why φ is not an algebra homomorphism. Then the deformed map $\tilde{\tau}$ is the map that "integrates out the fibers," giving an "effective action" on the classical vacua. Then the contour γ picks out a measure on the classical vacua, but only after you've chosen the fibration, which is as it is in the usual situation.

Concluding remarks

I'm sure I've already run out of time, so let me just mention a few more things.

Remark: It's essentially trivial to allow s lower-order terms.

Theorem: Using the same techniques, but under a much stronger genericity condition on s, you can uniformly (and highly non-canonically) choose the splitting φ. Namely, the three vector spaces above each have a basis consisting of the restrictions of the monomials $x_1^{m_1} \cdots x_n^{m_n}$ which are deparately of degree $< d - 1$ in each variable, i.e. $m_i < d - 1$ $\forall i$. Then you can define φ on this basis to be $\varphi([x_1^{m_1} \cdots x_n^{m_n}]) = x_1^{m_1} \cdots x_n^{m_n}$, and the isomorphisms are the identity for this basis.

Remarks on Chern–Simons theory: I mentioned the Volume Conjecture in the abstract, so let me say a few words. Certainly I have no results, just some directions to look. The Volume Conjecture predicts that certain limits of the Chern–Simons path integral are dominated by imaginary critical points — the Jones polynomial should be computed by an integral of the form $\int f\, e^s$ where s is the Chern–Simons functional, and we usually think of it as an integral over the space of SU(2) connections, but the dominant contribution in the limit in Volume Conjecture is from a flat $\mathrm{SL}(2,\mathbb{R})$ connection. A year or two ago Witten wrote a hundred-page paper exploring this and related mysteries.

However, the moral of the above work is that this isn't at all mysterious. s is a polynomial, and if f is polynomial (or polynomial times exp(quadratic)), then it should not be a surprise that all the scheme-theoretic critical points contribute. This is in stark contrast to the semiclassical/perturbative case, where the same techniques give you Feynman diagrams but the role of $\{\mathrm{d}s = 0\}$ is played by the smooth (real) critical locus. Now, Witten's paper has lots of other things in it — he considers $\mathrm{SL}(2,\mathbb{C})$ both as an algebraic object and as a smooth object. See, SU(2) and $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ are almost indistinguishable from the point of view of polynomial functions, but to smooth functions they look very different. So you should think about what types of functions you actually care about.

What I'd like to be able to do is use similar techniques to write down a homological definition of the non-perturbative Chern–Simons path integral. And perhaps verify that it satisfies the Jones Polynomial skein relation. What you'd have to do is to choose (you're in infinite dimensions, so there are lots of choices) some algebrogeometric version of the space of connections, and there are derived-algebraic-geometry ways of taking the quotient modulo gauge. Then you should work out what is the Koszul complex for the subspace of flat connections.

This can all be done, in too many ways. The hard part is to find any perturbation at all that deserves to be thought of as "doing the integral" — the $\sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}$ part of ∂_{full}. Presumably, if you have one, then the same game with $\mathbb{N}$-gradings will make it small. What gets in the way of just writing it down naively is the same thing that in the case of perturbative integrals causes *ultraviolet divergences*. For perturbative integrals, there is a well-understood way of trying to fix these, by working order-by-order in Planck's constant, but that's not available here. Now, maybe it can be done — in fact Chern–Simons theory doesn't really have UV divergences. I haven't tried too hard yet, but it's one of the directions to look.