A properadic approach to the deformation quantization of topological field theories

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Thank you, Peter, for the invitation. I will warn at the outset that, like in any good mathematics talk, I'm going to elide a whole lot of technical details. You can find them in `arXiv:1307.5812`.

My goal for this talk is to explain the following:

Lemma Any Poisson structure on the infinitesimal manifold $\mathrm{spec}(\widehat{\mathrm{Sym}}(V))$ induces a homotopy 1-shifted Lie bialgebra structure on V.

Definition A Lie bialgebra V with bracket $\beta : V \otimes V \to V$ and cobracket $\delta : V \to V \otimes V$ is *surinvolutive* if

$$\beta \circ (\mathrm{id} \otimes \beta) \circ (\delta \otimes \mathrm{id}) \circ \delta = 0 : V \to V. \tag{$*$}$$

Theorem There is a canonical wheel-free universal deformation quantization of Poisson infinitesimal manifolds whose induced homotopy Lie bialgebra structures are homotopically surinvolutive.

Before I continue, I have a question for all the Lie theorists in the room: Have you seen the composition in $(*)$? It's a souped-up version of involutivity, which is why I've used the name "surinvoulitivity," but I'd love to have a better name.

I should also say: I have a rough idea of how to prove that the theorem is sharp — that the Poisson structures that admit wheel-free deformation quantization are precisely the homotopy-surinvolutive ones. But I haven't written out the details yet.

It may not look it, but the theorem is completely combinatorial: at any given order, there's a finite combinatorial description of what is a "homotopy surinvolutive Poisson structure." Of course, I realize that I haven't given you the faintest clue how to compute such a description, or even what it should look like. I'll make the notion of "homotopy [. . .] structure" in terms of properads, which I will tell you about. First, I'll tell you a little about infinitesimal manifolds, and a little about topological field theory and deformation quantization.

Infinitesimal homotopy BD manifolds

Smooth manifolds have something to do with the real numbers $\mathbb{R}$ specifically. But you can do calculus with power series, and power series make sense over any field of characteristic 0. (They also make sense in non-zero characteristic, but you have more choices to make and funny things can happen.) So for this talk, I'll work over $\mathbb{Q}$.

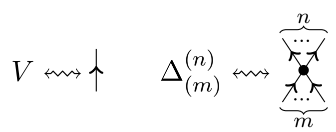
Definition An *infinitesimal manifold* is spec of a completed symmetric algebra $V = \mathrm{spec}(\widehat{\mathrm{Sym}}(V))$ for V any chain complex (V, ∂_V). (Aside: I use chain complexes a lot. My convention is that $\deg(\partial_V) = -1$.)

What data comprises a differential operator Δ on X? Assuming it's continuous for the power series topology, you can write it in coefficients:

$$\Delta = \sum_{m,n} \Delta_{(m)}^{(n)}$$

where $\Delta_{(m)}^{(n)}$ is the differential operator determined by a tensor $\mathrm{Sym}^m(V) \to \mathrm{Sym}^n(V)$. The sum over m truncates — it's the order qua differential operator. The sum over n is infinite — it's the Taylor expansion of the coefficients.

It's convenient to draw pictures of tensors; then contraction of tensors is contraction of diagrams:



Up to deciding on a convention for coefficients like $n!$ and so on, composition of differential operators is some sort of sum-over-diagrams activity.

One of the most important structures is the following:

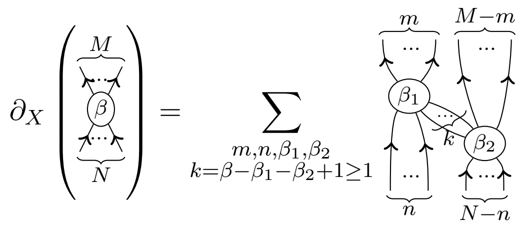
Definition A *semistrict homotopy BD structure* in X is $\Delta : \mathcal{O}(X) \to \mathcal{O}(X)[[\hbar]]$ of homological degree -1, such that:

1. $\Delta^2 = 0$
2. $\Delta(1) = 0$
3. Modulo $\hbar^N$, Δ is an Nth-order differential operator.

Before I continue, here's a little history. A *BD structure* is when 3. is replaced by: 3'. Δ is second-order as a differential operator. (Remember, Grothendieck says that a *Nth-order differential operator on X* is an endomorphism Δ of $\mathcal{O}(X)$ such that $[\Delta, f\cdot]$ is $(N-1)$th-order for all $f \in \mathcal{O}(X)$. In homological land, you need to take the graded commutator.) These structures appear in the Batalin–Vilkovisky approach to oscillating integrals and QFT. Unfortunately, the first mathematicians to really think about BV integrals were also thinking about the Deligne conjecture on Hochschild cohomology, where a very similar — but not the same — structure appears, and so in math "BV structure" means the wrong thing. (It's the same if you only use $\mathbb{Z}/2$ gradings.) Beilinson and Drinfeld seem to be the first mathematicians to really use the physics-BV gradings, and so Costello and Gwilliam have named it after them.

The word "semistrict" means "don't homotopize the Leibniz rule." For those in the know, here is an **Exercise:** Let P_N be the principal symbol of $\Delta \mod \hbar^N$. Then all together, the P_Ns define a flat L_∞ algebra structure on $\mathcal{O}(X)[-1]$.

If $X = \mathrm{spec}(\widehat{\mathrm{Sym}}(V))$, then we can expand Δ into a bunch of tensors. There's also the sum over powers in $\hbar$: $\Delta = \sum_{k\geq 1} \hbar^{k-1}\Delta_{;k}$, and then $\Delta_{;k} = \sum \Delta^{(n)}_{(m);k}$, with $m \leq k$ by 3.. I will actually index by $\beta = k - m$. Note that 2. is equivalent to $m \geq 1$. Assume also $n \geq 1$ (i.e. Δ vanishes at the origin). Then the term with $(m, n, \beta) = (1, 1, 0)$ is a linear differential on V, and I'll absorb it into ∂_V. All together, equation 2. then reads:



(**)

See, β is playing the role of "internal genus of the vertex," and equation (**) is homogeneous for "total genus" = genus of diagram + sum of internal genera.

A one-dimensional topological field theory

Here's what they're good for. Suppose you give $\mathrm{Chains}(\mathbb{R}) \otimes V$ a hBD structure (choose your favorite model of $\mathrm{Chains}(\mathbb{R})$). Suppose also that there's some *quasilocality* built in: for each (m, n, β), the corresponding tensor $(\mathrm{Chains}(\mathbb{R}) \otimes V)^{\otimes m} \to (\mathrm{Chains}(\mathbb{R}) \otimes V)^{\otimes n}$, when you think in terms of "integral kernels" in $\mathbb{R}^{m+n}$, is supported in a finite-radius tubular neighborhood of the diagonal $\mathbb{R} \hookrightarrow \mathbb{R}^{m+n}$.

You can, of course, choose deformation retractions of $\mathrm{Chains}(\mathbb{R})\otimes V$ onto V, since $\mathrm{H}_\bullet(\mathrm{Chains}(\mathbb{R})) = \mathbb{Q}$ — the inclusion $V \to \mathrm{Chains}(\mathbb{R}) \otimes V$ requires choosing a point in $\mathbb{R}$ (or a bump function, or ...). You can extend this retraction to a deformation retraction of $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}(\mathbb{R}) \otimes V\big)$ onto $\widehat{\mathrm{Sym}}(V)$. Of course, the hBD structure Δ is a *perturbation* of the linear differential, and so homotopy perturbation theory gives you also deformation retractions of $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}(\mathbb{R})\otimes V\big)[\![\hbar]\!]$ with differential $\partial_{\mathrm{dR}} + \partial_V + \Delta$ onto some complex with underlying graded vector space $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$, and I'll assume that V is concentrated in degree 0. This deformed retraction still depends on choosing a point $z \in \mathbb{R}$. I'll call the deformed inclusion "insertion at z", and the projection "expectation value."

So, let's choose two functions $f_1, f_2 \in \widehat{\mathrm{Sym}}(V)$, and insert them at $z_1, z_2 \in \mathbb{R}$. You can multiply their images in $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}(\mathbb{R}) \otimes V\big)[\![\hbar]\!]$ — it's not a dg algebra, but it is an algebra. Then calculate the expectation value of the product. This is the *two-point function*.

You can write out explicit formulas for insertion and expectation values, by looking up Crainic's paper on the homological perturbation lemma. The operator Δ appears in the formulas. As such, when z_1 and z_2 are "close" to each other, they "interact" in Δ, and so there's no particular reason for the two-point function to be independent of z_1, z_2. On the other hand:

Proposition Quasilocality implies that the limit as $z_2 - z_1 \to \infty$ of the two-point function converges in the power series topology.

Again, there's no reason why the value of the two-point function when $z_2 \gg z_1$ should agree with the value when $z_1 \gg z_2$. Well, there is something:

Proposition In the limit as $z_2 \gg z_1$, the two-point function is an associative deformation of the commutative product on $\widehat{\mathrm{Sym}}(V)$.

This is because associative multiplication is about the topology of $\mathbb{R}$. Note that there's no reason why the two-point function should be commutative.

Properads

So there's a good reason to care about hBD structures on infinitesimal manifolds. The question now is: how do you construct interesting hBD structures? To make this more precise, there is a useful technical tool, called *properads*, which control differential operators acting on infinitesimal manifolds.

Definition A *properad* P is a chain complex $P(m,n)$ for each $m,n \in \mathbb{N}$ of "m-to-n operations," along with "composition" maps for each connected directed acyclic graph.

Examples There is a properad $\mathrm{End}(V)$ for any chain complex V, with $\mathrm{End}(V)(m,n) = \hom(V^{\otimes m}, V^{\otimes n})$. There is a properad hBD such that hBD structures on V are the same as homomorphisms $\mathrm{hBD} \to \mathrm{End}(V)$ — it is freely generated by degree-(-1) elements $\gamma(m,n,\beta)$ for each m,n,β, with differential given by equation $(**)$.

There is a properad $\mathrm{QLoc} \subseteq \mathrm{End}(\mathrm{Chains}(\mathbb{R}))$ with $\mathrm{QLoc}(m,n)$ consisting of the operations whose integral kernels are supported in a finite-radius neighborhood of the diagonal. A quasilocal hBD structure on $\mathrm{Chains}(\mathbb{R}) \otimes V$ is the same as a homomorphism $\mathrm{hBD} \to \mathrm{QLoc} \otimes \mathrm{End}(V)$.

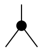
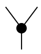
Here are a few more:

Definition A *open and coopen d-shifted commutative Frobenius algebra* is a chain complex V with a commutative non-unital ("open") multiplication on $V[-d]$ and a cocommutative non-counital ("coopen") comultiplication on V. (I.e. multiplication has degree $-d$ and comultiplication has degree d.) A typical example is $\mathrm{H}_\bullet(M)$ for M a d-dimensional oriented manifold.

Exercise Write this as a properad Frob_d.

Definition The properad $\mathrm{invFrob}_d$ also imposes the relation multiplication $\circ$ comultiplication $= 0 : V \to V$. Thus $\mathrm{invFrob}_d(m,n) = \mathbb{Q}$ if $m,n \geq 1$.

The properads Frob_d, $\mathrm{invFrob}_d$, and hBD each have an extra grading by "genus."

Definition There is a properad LB_d of Lie bialgebras in which the cobracket has degree -1 and the bracket has degree $d-1$. The reason for this convention on degrees is that $\mathrm{invFrob}_d$ is *quadratic* — it is generated by  and  with only quadratic relations — and its "quadratic dual" (think Koszul duality!) is LB_d.

The gradings are such that a usual Lie bialgebra structure on V is the same as an LB_2 structure on $V[-1]$.

There's an important construction on properads:

Definition Let P be a properad, satisfying some mild finite-dimensionality conditions that I'll leave out. The *Bar dual* of P is the properad $\mathbb{D}\,P$ freely generated by $P^*[-1]$ with differential dual to binary composition in P.

Important fact Under mild conditions, $\mathbb{D}\,\mathbb{D}\,P \to P$ is a cofibrant replacement. A *homotopy P*-algebra structure on V is a homorphims $\mathrm{h}P \to \mathrm{End}(V)$ for any cofibrant replacement $\mathrm{h}P \to P$. (Axioms of model categories assure that it doesn't matter which cofibrant replacement you use.)

This is why $\mathbb{D}$ is a type of "duality." $\mathbb{D}^2$ isn't the identity, but it's canonically homotopic to the identity on cofibrant properads.

Example $\mathrm{hBD} = \mathbb{D}\,\mathrm{Frob}_0$.

Homotopy Lie Bialgebra structures and deformation quantization

It turns out that $\mathrm{invFrob}_d$ and LB_d are Koszul. (N.B.: It is currently an open question whether Frob_0 is or is not Koszul.) What this means is that $\mathbb{D}\,\mathrm{invFrob}_d \to \mathrm{LB}_d$ is a cofibrant replacement. ($\mathbb{D}\,P$ is always cofibrant, and $\mathbb{D}\,\mathrm{invFrob}_d \to \mathrm{LB}_d$ is a fibration by a trivial argument; it's the fact that it's acyclic that's hard, and more or less requires the theory of Lyndon words.)

What is $\mathbb{D}\,\mathrm{invFrob}_d$? It's freely generated by a generator $\pi^{(n)}_{(m)}$ with m inputs and n outputs for each m, n, with $\partial(\pi^{(n)}_{(m)})$ = a sum over genus-zero diagrams with two vertices.

Exercise Suppose given a homomorphism $\mathbb{D}\,\mathrm{invFrob}_d \to \mathrm{End}(V)$. Turn $\sum_n \pi^{(n)}_{(m)}$ into a multiderivation $\pi_{(m)}$ in m variables on $\widehat{\mathrm{Sym}}(V)$. Then the $\pi_{(-)}$s together give $\widehat{\mathrm{Sym}}(V)[d-1]$ an L_∞ algebra structure.

I.e.: the infinitesimal manifold $X = \mathrm{spec}\,\widehat{\mathrm{Sym}}(V)$ has a homological vector field $\pi_{(1)}$, a bracket $\pi_{(2)}$ which satisfies a Jacobi identity up to the differential of $\pi_{(3)}$, and so on.

Lemma from beginning of talk Suppose that V is concentrated in degree 0, and let $d = 1$. Then only $\pi_{(2)}$ is non-zero, for degree reasons. Since $\pi_{(3)} = 0$, and $[\pi_{(2)}, \pi_{(2)}] = [\pi_{(1)}, \pi_{(3)}] = 0$, this bivector field $\pi_{(2)}$ is a Poisson structure on X. Conversely, any Poisson structure on X gives a $\mathbb{D}\,\mathrm{invFrob}_d$ structure on V. Don't forget: $\mathbb{D}\,\mathrm{invFrob}_1$ is a cofibrant replacement for LB_1.

More generally, a $\mathbb{D}\,\mathrm{invFrob}_d = \mathrm{hLB}_d$ structure is a *semistrict homotopy* Pois_d *structure*, so that Poisson = Pois_1.

So, I asked before: how can you find quasilocal hBD structures on $\mathrm{Chains}(\mathbb{R}) \otimes V$? You can ask the same question for other manifolds. In properad language, the question is: how can you find homomorphisms $\mathrm{hBD} \to \mathrm{QLoc} \otimes \mathrm{End}(V)$?

In particular, are there structures on QLoc and $\mathrm{End}(V)$ independently that will assure such a homomorphism? The answer is yes:

Proposition Let P be a genus-graded properad satisfying mild conditions. Then there is a canonical homomorphism $\mathrm{hBD} \to \mathbb{D}P \otimes P$. **Corollary** Suppose that you have constructed a universal quasilocal P action on $\mathrm{Chains}(\mathbb{R})$. Then any $\mathbb{D}P$ algebra V determines a quasilocal hBD structure on $\mathrm{Chains}(\mathbb{R}) \otimes V$.

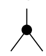
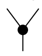
With this, I can almost prove the existence of universal deformation quantization:

False theorem There is a universal wheel-free deformation quantization. (Kontesvich proved a true theorem which is the same without the words "wheel-free." See, his quantization requires taking traces of the Taylor coefficients of the Poisson structure. "Wheel-free" means "only use acyclic graphs," which is imposed by working with properads and not "wheeled properads.")

False proof Since $\mathbb{R}$ is a one-dimensional oriented manifold, chain-level Poincare duality determines a non-trivial homotopy $\mathrm{invFrob}_1$ structure on $\mathrm{Chains}(\mathbb{R})$. (To make this precise, you really should work both with $\mathrm{Chains}(\mathbb{R})$ and $\mathrm{Cochains}(\mathbb{R})$, and the inclusion of chains as compactly supported cochains.) Involutivity is just because when d is odd, $\mathrm{invFrob}_d = \mathrm{Frob}_d$, because a commutative multiplication on $V[-d]$ is anticommutative on V.

Thus we have a map $\mathbb{D}\mathbb{D}\,\mathrm{invFrob}_1 \to \mathrm{QLoc}$, and thus if V is any $\mathbb{D}\,\mathrm{invFrob}_1$ algebra, and in particular if $\mathrm{spec}(\widehat{\mathrm{Sym}}(V))$ is any Poisson infinitesimal manifold, then we get a quasilocal hBD structure on $\mathrm{Chains}(\mathbb{R}) \otimes V$. A calculation proves that $\lim_{z_2 - z_1 \to \infty}$(two-point function) is a deformation in the direction of the Poisson bracket. □

In fact, false theorem really is false — this follows from calculations of Penkava and Vanhaecke; details are in a very recent paper by Dito. Actually, I claimed essentially the above proof in a paper at the beginning of this summer, because I thought I had constructed a homomorphism $\mathbb{D}\mathbb{D}\,\mathrm{invFrob}_1 \to \mathrm{QLoc}$. Then Dito and Merkulov and Willwacher and I all tried to figure out if my claimed theorem was possibly correct, because the preponderance of evidence suggested that wheel-free universal deformation quantization doesn't exist. Finally, we all roughly simultaneously found the proof that wheel-free universal quantization doesn't exist, and I found the error in my construction.

I should say, I didn't really work with $\mathbb{D}\mathbb{D}\,\mathrm{invFrob}_1$. Rather, since LB_1 is Koszul, I worked with $\mathbb{D}\,\mathrm{LB}_1$, which is also a cofibrant replacement for $\mathrm{invFrob}_1$, but is much smaller. The requirement in order for the proof to work is that the generators dual to the brackets  and  get mapped to Thom forms under the homomorphism $\mathbb{D}\,\mathrm{LB}_1 \to \mathrm{QLoc}$. Then the generators dual to binary compositions all have vanishing obstructions, since the composition of Thom forms is a Thom form, and the difference between two Thom forms is exact. It turns out moreover that the obstructions to represent the genus-one generators also vanish in homology. But if you do the calculations correctly (as I failed to do initially), there is a non-zero genus-two obstruction, which corresponds exactly to the composition in equation (∗) that I originally asked you about.

But obstruction theory does prove:

Theorem Let $\mathrm{surinvLB}_1$ denote the genus-graded properad given by quotienting LB_1 by the properadic ideal generated by the composition from equation (∗), so that actions of $\mathrm{surinvLB}_1$ are the same as 1-shifted surinvolutive Lie bialgebras. Then there exists a map $\mathbb{D}\,\mathrm{surinvLB}_1 \to \mathrm{QLoc}$ such that the  and  are mapped to Thom forms — in fact, the space of all such maps has the homotopy type of a point.

Corollary (Theorem from beginning of talk) Any $\mathbb{D}\mathbb{D}\,\mathrm{surinvLB}_1$ algebra (i.e. homotopy surinvolutive 1-shifted Lie bialgebra) admits a canonical wheel-free deformation quantization.

Let me conclude with a few optimistic generalizations. Suppose you replace $\mathbb{R}$ with $\mathbb{R}^d$. Then you can still talk about quasilocal operations, and try to play the same game. In particular, if you can construct a quasilocal homotopy $\mathrm{invFrob}_d$ structure on $\mathrm{Chains}(\mathbb{R}^d)$, then you'd get "a universal E_d quantization of semistrict homotopy Pois_d algebras." When $d \geq 2$, which is essentially equivalent to the formality of the E_d operad. So I conjecture that such structures do exist. Unfortunately, the straightforward obstruction theory isn't good enough: when $d \geq 2$, there are infinitely many obstructions that might not automatically vanish by degree reasons, and I don't know how to check all of them.

Here's a special case. The problem when $d = 2$ is very closely related to Etingof–Kazhdan quantization of usual Lie bialgebras. (What I'm saying now will hopefully be joint work with Owen Gwilliam, if we can make it work.) The reason is that you consider field theories on the horizonal strip, with boundary conditions on the edges, and this makes a Hopf algebra: multiplication is horizontal pasting, and comultiplication is cutting down the middle of the strip.

So I conjecture that there exists a quasilocal homotopy $\mathrm{invFrob}_2$ structure on $\mathrm{Chains}(\mathbb{R}^2)$. Moreover, I conjecture that the space of such structures is canonically homotopy equivalent to the space of Drinfeld associators. But these are hard conjectures. They have practical import: the obstruction theory implies, for example, that the space of maps $\mathbb{D}\,\mathrm{LB}_2 \to \mathrm{QLoc}(\mathbb{R}^2)$ is a disjoint union of classifying spaces of pronilpotent groups (given the pronilpotent topology: each finite-dimensional quotient should be treated as a discrete group).

So for example, if you know that the space of Drinfeld associators has higher homotopy groups, please let me know, because then I won't try to prove the conjecture.