A salad of BV integrals and AKSZ field theories, served over a bed of properads; it comes spiced with chain-level Poincare duality and just a pinch of Poisson geometry

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Thank you Jonathan for the invitation. And I was told these talks are supposed to be understandable and accessible. So please interrupt me with questions, because otherwise they will be understandable and accessible only to the speaker. My talks are based on a couple papers: `arXiv:1202.1554` with O. Gwilliam, and `arXiv:1307.5812`, which is very new, and the references therein.

Before I begin, let me give an over-ambitious outline:

1. **Oscillating integrals, homological perturbation theory.** This part should be "elementary," although my goal will be to motivate the homological perturbation lemma (which is a version of spectral sequences) from the problem of computing expectation values of observables in path integrals.
2. **Feynman diagrams, BV geometry, properads.** Properads are like associative algebras, except in an algebra you multiply in a line, and in a properad you multiply in a directed acyclic graph. One reason for talking about these is because Feynman diagrams are already a kind of graph. I won't talk about much geometry, but I'll give some definitions of geometric structures and I'll assert that they're important.
3. **Poisson AKSZ theory.** The last part, if I get to it, is my real motivation. I'll mention some conjectures and results about deformation quantization and topological field theories.

1 Oscillating integrals, homological perturbation theory

For me, the start of the whole story is a proposal by Feynman, which is that quantum physics has something to do with calculating the following integrals:

$$\langle f \rangle = \frac{\displaystyle\int f \, \exp\left(\frac{i}{\hbar} s\right) \mathrm{dVol}}{\displaystyle\int \exp\left(\frac{i}{\hbar} s\right) \mathrm{dVol}}$$

The integral is supposed to range over some, often infinite-dimensional, space. Volume forms on infinite-dimensional manifolds are hairy, although Jonathan, for example, has some ways of doing them. $\hbar$ is called "Planck's constant," and is "small." $i = \sqrt{-1}$. s is the "action," which is known (or guessed) and controls the "physics." f is the "observable," and $\langle f \rangle$ is its "expectation value," which is what you actually measure in "experiments."

As a mathematician, of course, I don't care what any of these words actually mean. But integrals are things I do care about. Let's make it easy and integrate over $\mathbb{R}^n$, and make $\mathrm{dVol} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ the standard measure. For now, let's not decide what type of functions we're integrating. And let's just not worry very much about issues of convergence. Physicists like to calculate now, justify later. Also, I'm going to absorb i into $\hbar$, which is just a variable anyway, and replace $\frac{i}{\hbar}$ with $\frac{1}{\hbar}$. Maybe this means that $\hbar$ is now "pure imaginary."

So, let's see. dVol is translation invariant — this is its defining property — meaning that we have an "integration by parts" formula. This means that:

$$\int \frac{\partial}{\partial x_i} \left(g\, e^{s/\hbar} \right) = \text{boundary term}$$

Let's assume that the boundary term vanishes. Then

$$0 = \int \frac{\partial}{\partial x_i} \left(g e^{s/\hbar} \right) = \int \frac{\partial g}{\partial x_i}\, e^{s/\hbar} + \frac{1}{\hbar} \int g\, \frac{\partial s}{\partial x_i}\, e^{s/\hbar}$$

Put another way:

$$\left\langle \hbar \frac{\partial g}{\partial x_i} + g\, \frac{\partial s}{\partial x_i} \right\rangle = 0, \qquad \text{i.e.} \quad \left\langle g\, \frac{\partial s}{\partial x_i} \right\rangle = -\hbar \left\langle \frac{\partial g}{\partial x_i} \right\rangle$$

Any equation like this is called a *Ward identity.* They're pretty awesome: with some luck, if you write down a whole lot of Ward identities, and then solve the system of equations, you get an answer in terms of $\langle 1 \rangle = 1$.

But it's still a mess. A general organizing principal is: *package up messes as chain complexes.* See, we have n maps of the form $g \mapsto g\, \frac{\partial s}{\partial x_i} + \hbar \frac{\partial g}{\partial x_i}$, and $\langle \cdot \rangle$ vanishes on the images of all of them. Let's write $\mathcal{O}$ for the algebra of observables. Then the Ward identities are equivalent to saying that $\langle \cdot \rangle$ vanishes on the image of the map:

$$\mathcal{O}^{\oplus n} \xrightarrow{\begin{pmatrix} \frac{\partial s}{\partial x_1} + \hbar \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial s}{\partial x_n} + \hbar \frac{\partial}{\partial x_n} \end{pmatrix}} \mathcal{O} \tag{∂}$$

Think of this as a two-term chain complex in degrees 1 and 0. Then the point is that $\langle \cdot \rangle$ is actually a map out of H_0 of this complex.

Hope: H_0 of this complex is something small. At best, it's one-dimensional, and then identifying $[1] \in \mathrm{H}_0$ with $1 \in \mathbb{C}$ gives the map $\langle \rangle$.

In fact, this hope depends a lot on what types of functions are in $\mathcal{O}$.

But we also want a procedure for actually calculating $\langle f\rangle$. So let's make some assumptions. First, let's assume $\hbar \ll 1$, and be satisfied calculating $\langle f\rangle$ as an element of $\mathbb{C}[[\hbar]]$, meaning we're satisfied if we can calculate $\langle f\rangle \bmod \hbar^N$ for large N.

Look back at the Ward identity. Recall that the *critical locus of s* is the symultaneous vanishing locus of $\frac{\partial s}{\partial x_1}, \dots, \frac{\partial s}{\partial x_n}$. Then the Ward identity says: "Suppose f vanishes on the critical locus of s. Then $\langle f\rangle = O(\hbar)$." This is the *stationary phase approximation.*

For simplicity, let's suppose that s has a critical point at the origin. Then:

$$s(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_i x_j + b(x)$$

where a_{ij} is the Hessian of s at 0, and $b(x)$ is divisible by third-powers in the x_is. Let's also suppose that this is an isolated critical point, so that the matrix a is invertible.

Exercise: Prove that if $f(0) = 0$, then there exist $g_1, \dots, g_n$ so that in a neighborhood of the origin, $f(x) = \sum_i g_i(x)\,\frac{\partial s}{\partial x_i}$.

Actually, suppose we can find formulas for the g_is. Then for any f, we'd have:

$$\langle f\rangle = \langle f(0)\rangle + \langle f - f(0)\rangle = \langle f(0)\rangle + \sum_i \left\langle g_i(x)\,\frac{\partial s}{\partial x_i}\right\rangle = \langle f(0)\rangle - \hbar\sum_i \left\langle \frac{\partial g_i}{\partial x_i}\right\rangle, \qquad (*)$$

modulo contributions from outside of the neighborhood of the origin.

Actually, let's just assume that these contributions go away, and declare that $\mathcal{O} = \mathbb{C}[[x_1, \dots, x_n]]$, so that we're working in formal power series. This will also help later.

Second, what type of data is the n-tuple $(g_1, \dots, g_n)$? It's a vector that depends on x, i.e. a *vector field.* Actually, we should have recognized this back when we wrote down our two-term chain complex: $\mathcal{O}^{\oplus n} = \mathcal{T}$ is the space of vector fields. Then $\sum_i g_i \frac{\partial s}{\partial x_i}$ is nothing but the action of $\vec{g}$ on s — vector fields act as derivations — and $\sum_i \frac{\partial g_i}{\partial x_i}$ is nothing but the divergence of $\vec{g}$. Actually, the whole differential is just ($\hbar$ times) the divergence with respect to $e^{s/\hbar}\mathrm{dVol}$, so it's coordinate-independent.

Let's return to the exercise, and find $\vec{g}$ provided we're working in formal power series. We have Taylor's expansion:

$$f(x) = \sum f_k\, x_k + \frac{1}{2}\sum f_{kl}\, x_k x_l + \dots \stackrel{?}{=} \sum g_i(x)\, a_{ij}x_j + \sum g_i(x)\,\frac{\partial b(x)}{\partial x_i}$$

And a_{ij} is invertible, and $\frac{\partial b(x)}{\partial x_i}$ vanishes at least second order. Let's call $\eta = a^{-1}$. Then, dropping the $\sum$ signs, we have:

$$f(x) = f_k \eta_{ki} a_{ij} x_j + \frac{1}{2}\sum f_{kl}\, x_k x_l + \dots$$

So we should set $g_i(x) = \sum_k f_k \eta_{ki} + O(x)$. Then to calculate the next term, we can assume that f is quadratic:

$$\frac{1}{2}\sum f_{kl}\, x_k x_l + \cdots = (f_{kl} x_k \eta_{li}) a_{ij} x_j + \ldots$$

So for such an f, we could choose $g_i(x) = \sum_{kl} f_{kl} x_k \eta_{li} + \ldots$.

Rather than working out what's going on with the bs, which always raise degree in x, let's revise $(*)$. For any $f \in \mathcal{O} = \mathbb{C}[\![x_1, \ldots, x_n]\!]$, let's define $\eta(f) = (\eta_1(f), \ldots, \eta_n(f)) \in \mathcal{T}$ so that $f = f(0) + \sum_{ij} \eta_i(f)\, a_{ij} x_j$. Here's one way to do this: declare that if f is homogeneous of degree $N \geq 1$, then

$$\eta_i(f) = \frac{1}{N}\frac{\partial f}{\partial x_i}$$

and $\eta(\text{constant}) = 0$.

With this in hand, we have:

$$\langle f \rangle = \langle f(0) \rangle + \left\langle \sum_{ij} \eta_i(f)\, a_{ij} x_j \right\rangle = \langle f(0) \rangle + \left\langle \sum_i \eta_i(f)\, \frac{\partial s}{\partial x_i} \right\rangle - \left\langle \sum_i \eta_i(f)\, \frac{\partial b}{\partial x_i} \right\rangle =$$

$$= \langle f(0) \rangle - \hbar \left\langle \sum_i \frac{\partial}{\partial x_i} \eta_i(f) \right\rangle - \left\langle \sum_i \eta_i(f)\, \frac{\partial b}{\partial x_i} \right\rangle \quad (**)$$

Does this help simplify things? Yes: the middle term is higher-order in $\hbar$, and the last term is higher-order in x (since η drops degree in x by 1, but $\frac{\partial b}{\partial x_i}$ is degree at least 2 in x). So we can just keep repeating this, and as long as everything is continuous for the power series topology on $\mathcal{O}[\![\hbar]\!] = \mathbb{C}[\![x_1, \ldots, x_n, \hbar]\!]$, then we win.

Let's try to understand $(**)$ in relation to the little chain complex (∂). See, we had a pretty complicated map $\partial : \mathcal{T} \to \mathcal{O}$. But it came in some pieces:

$$\partial = a + \delta$$

where

$$a(\vec{g}) = \sum_{ij} a_{ij}\, g_i(x)\, x_j$$

$$\delta(\vec{g}) = \sum_i \left(g_i(x) \frac{\partial b}{\partial x_i} + \hbar \frac{\partial g_i}{\partial x_i} \right)$$

It's immediately clear that, thinking of a as a differential for its own chain complex, then $\mathrm{H}_0\big(\mathcal{T} \overset{a}{\to} \mathcal{O}\big) = \mathbb{C}$. Indeed, that's what η witnesses. Let $p : \mathcal{O} \to \mathbb{C}$ be $p(f) = f(0)$, and $\iota : \mathbb{C} \to \mathcal{O}$ is the canonical inclusion (in particular, $p \circ \iota = \mathrm{id}$). Then $\eta : \mathcal{O} \to \mathcal{T}$ is a homotopy such that $\iota \circ p = \mathrm{id} - [a, \eta]$ in degree 0, where $[a, \eta] = a \circ \eta + \eta \circ a$, since both shift homological degrees by an odd amount. Of course, I'm ignoring right now the homology in degree 1, but I'll come back to it.

Then δ is a *perturbation* of the differential, turning a into ∂. Iterating $(**)$ says that after this perturbation, we can get a new projection $\tilde{p} : \mathcal{O} \to \mathbb{C}[\![\hbar]\!]$, which works for the differential ∂ rather than for a, given by the formula:

$$\tilde{p} = p + p \circ (-\delta \circ \eta) + p \circ (-\delta \circ \eta) \circ (-\delta \circ \eta) + \cdots = p \circ (\mathrm{id} + \delta\eta)^{-1}$$

And this converges since $\delta\eta$ raises degree, and so is small for the power series topology.

Well, it almost says this. What we've actually proven is that *if* $\mathrm{H}_0\big(\mathcal{T} \overset{a+\delta}{\to} \mathcal{O}\big) \cong \mathbb{C}[\![\hbar]\!]$, then the projection $\mathcal{O} \to \mathrm{H}_0$ is given by $\tilde{p}$. Equivalently, *if* the original oscillating integral $\langle f \rangle$ has an asymptotic expansion as $\hbar \to 0$, then it's given by the above calculations. We haven't actually proven that the homology for the perturbed differential isn't much smaller. This has practical importance: what if you and I go and applying Ward identities in different ways? Can we get different answers? Maybe there's just no asymptotic expansion. We certainly didn't touch the analytic estimates necessary to verify *that*.

What we need to know is that the *closed* elements of $\mathcal{T}$ for the differential a don't all of a sudden start messing things up. Of course, if you're closed for a, you won't be closed to ∂. But since $\partial = a + o(1)$, it seems like the closed guys should just move a little bit?

Well, who's a-closed? Say $n = 2$ and $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Consider $\vec{g} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = (-x_2, x_1)$. Then $a(\vec{g}) = -x_2 x_1 + x_1 x_2 = 0$. In general, the closed guys are precisely the image of the $\mathcal{O}$-linear map $\mathcal{T} \wedge_{\mathcal{O}} \mathcal{T} \to \mathcal{T}$ given on a basis by $\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \mapsto \sum_k a_{ki} x_k \frac{\partial}{\partial x_j} - \sum_k a_{kj} x_k \frac{\partial}{\partial x_i}$. So let's also call this a, and extend the complex:

$$\mathcal{T} \wedge \mathcal{T} \overset{a}{\to} \mathcal{T} \overset{a}{\to} \mathcal{O}$$

Then there's no homology in degree 1, which you can witness by finding a homotopy $\eta : \mathcal{T} \to \mathcal{T} \wedge \mathcal{T}$. Does the perturbation δ extend? Yes. There is such a thing as the *divergence* of a bivector field, and also you can contract a bivector field with the one-form $\mathrm{d}b$ to get a vector field.

Actually, we really ought to go all the way and work with multivector fields, since there's currently homology in degree 2. So our whole complex will be $\bigwedge^\bullet \mathcal{T}$. It's a graded commutative algebra, just like the de Rham complex is. It's convenient to change notation slightly, and let ξ_i denote the variable *in homological degree* 1 corresponding to $\frac{\partial}{\partial x_i}$. Then:

$$\textstyle\bigwedge^\bullet \mathcal{T} = \mathbb{C}[\![\xi_1, \dots, \xi_n, x_1, \dots, x_n]\!] = \widehat{\mathrm{Sym}}\big(\wr \oplus \uparrow\big)$$

where I've changed notation so that $\wr$ now means the vector space spanned by the ξ_is, so it's in homological degree 1. So symmetric powers of it are really wedge powers, because of the sign rules of chain complexes.

The simple differential a can now be written as:

$$a = \sum_{ij} a_{ij}\, x_i\, \frac{\partial}{\partial \xi_j}$$

and the perturbation is:

$$\delta = \sum_i \left(\frac{\partial b(x)}{\partial x_i} \frac{\partial}{\partial \xi_i} + \hbar \frac{\partial^2}{\partial x_i \partial \xi_i} \right)$$

The simple differential even has a simpler definition. The graded vector space $\wr \oplus \uparrow$ is in degrees 1 and 0, and since the matrix a is invertible, we can make it into an exact complex $\wr \xrightarrow{a} \uparrow$. Then the whole thing with the simple differential a is what you get by applying the "completed symmetric algebra" functor to this exact complex. In characteristic 0, taking symmetric powers is an exact functor. So this proves that the complex $\left(V = \bigwedge^\bullet \mathcal{T}, \partial_V = a\right)$ has as its homology just $\widehat{\mathrm{Sym}}(0) = \mathbb{C}$.

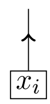
And if we try to turn on the differential, we can use the formula $\tilde{p}$ that we already have. We arrive at the following basic fact, discovered in the 70s by people trying to understand spectral sequences:

Homological Perturbation Lemma: A *retraction* of a chain complex (V, ∂_V) onto another chain complex (H, ∂_H) consists of chain maps $\iota : H \to V$ and $p : V \to H$ satisfying $p \circ \iota = \mathrm{id}$, and a homotopy $\eta : V_\bullet \to V_{\bullet+1}$ satisfying $\iota \circ p = \mathrm{id} - [\partial_V, \eta]$. (Signed commutator!) A *deformation* of (V, ∂_V) is a map $\delta : V_\bullet \to V_{\bullet-1}$ satisfying $(\partial_V + \delta)^2 = 0$. A deformation is *small* with respect to a retraction if $\mathrm{id} + \delta\eta : V \to V$ is invertible (this is a map of graded vector spaces, not complexes!). If it is, then the whole retraction deforms: you get new maps $\tilde{p} = p(\mathrm{id} + \delta\eta)^{-1}$, $\tilde{\iota} = \iota - \eta(\mathrm{id} + \delta\eta)^{-1}\delta\iota$, $\tilde{\eta} = \eta(\mathrm{id} + \delta\eta)^{-1}$, and $\tilde{\delta} = p(\mathrm{id} + \delta\eta)^{-1}\delta\iota : H \to H$, which together give a retraction of $(V, \partial_V + \delta)$ onto $(H, \partial_H + \tilde{\delta})$.

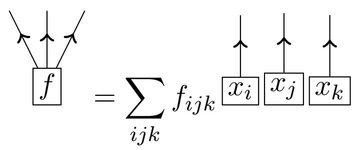
Proof: Exercise. It's pure algebraic manipulation.

2 Feynman diagrams, BV geometry, properads

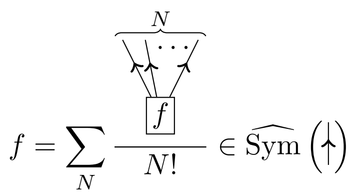
Equation $(**)$ also has a diagrammatic meaning. Let's draw a vertical line $\uparrow$ for the vector space with basis the variables $x_1, \dots, x_n$. (So this is the dual vector space to the $\mathbb{R}^n$ that we're integrating over.) We'll put things next to each other for tensor products, and we'll contract edges when we want to contract indices, i.e. compose tensors. This is a notation developed by Penrose. Then for example the element $x_i \in \uparrow$ would be written as:



Let's embed symmetric tensors among all tensors. And let's agree to draw the tensor $\sum f_{ijk} x_i x_j x_k$ as:



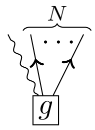
and more generally the Nth Taylor coefficient of f as a box with an f in it and N edges coming out. Then



and the $N!$ counts the number of symmetries of the diagram.

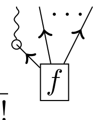
How should we draw a vector field like $\vec{g}$ of $\eta(f)$? Let's agree to use a wavy edge to denote the vector space with basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Then the Nth Taylor coefficient of $\vec{g}$ is really a tensor in $\cdots$ with N normal edges. So

$$g = \sum_N \frac{\qquad}{N!}$$



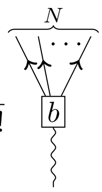
What does the homotopy $\eta : \mathcal{O} \to \mathcal{T}$ do? On a symmetric tensor with only edges, it takes the first (say) edge and applies the matrix $a^{-1} : \ \to$. For a more general tensor, it averages over all the choices of a edge to choose.

$$\eta(f) = \sum_{N \geq 1} \frac{1}{N!}$$

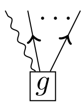


The one-form $\frac{\partial b}{\partial x}$, on the other hand, is a map $\ \to \widehat{\mathrm{Sym}}(\ )$:

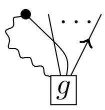
$$\frac{\partial b}{\partial x} = \sum_N \frac{1}{N!}$$



And finally, there's the divergence operation, which acts on g via:



$$\mapsto \sum_{i=1}^{N}$$



where the dot attaches to the ith strand out of N total.

Exercise: Drop some of the diagrammatic decorations. By iterating $(**)$, conclude that $\langle f \rangle$ is given by a "sum of Feynman diagrams," namely the sum of closed connected diagrams with one f vertex and lots of b vertices, weighted by $\hbar^\beta$ where β is the first Betti number of the diagram, and also weighted by the number of automorphisms of the diagram; and maybe there are some signs to work out too.

I bring this up because it emphasizes the following: *differential operators acting on completed symmetric algebras are coordinated by directed acyclic graphs.* To really emphasize this, let's work just with the complex $= \ \overset{a}{\to}$. Then all of our graphs above are to be read from bottom to top, even if I don't draw arrows.

So we have two complexes. We have $A = \widehat{\mathrm{Sym}}(\uparrow)$, with its differential a, and we have a deformation $A[\![\hbar]\!] = \widehat{\mathrm{Sym}}(\uparrow \oplus \hbar)$ with the differential $\partial = a + \delta$. This deformed complex is a graded commutative algebra, but not a dg algebra. The differential $\partial = a + \delta$ is a derivation modulo $\hbar$. But its order-$\hbar$ term is $\hbar \sum \frac{\partial^2}{\partial \xi_i \partial x_i}$, which is second-order. Its *principal symbol* is the biderivation $(f_1, f_2) \mapsto \partial(f_1 f_2) - (f_1 \partial f_2 + f_2 \partial f_1)$, with some signs thrown in if the f_i are odd. This is actually a version of a "Poisson bracket." A *Poisson bracket* on an algebra A is a biderivation making A into a Lie algebra. What we have is a biderivation making $A[-1]$ into a dg Lie algebra. For some reason, I will call this new thing "Pois_0," and the old thing "Pois_1."

Exercise: Suppose that A is a graded commutative algebra, with a differential ∂ on $A[\![\hbar]\!]$ which is second-order, but first-order modulo $\hbar$. Then $P : (f_1, f_2) \mapsto \hbar^{-1}\big(\partial(f_1 f_2) - (f_1 \partial f_2 + f_2 \partial f_1)\big)$ exists and makes A into a Pois_0 algebra.

This is the structure proposed by Batalin and Vilkovisky, and so ought to be called a "BV algebra." Actually, Batalin–Vilkovisky asked also that the Poisson structure be "symplectic" in an appropriate sense, but we won't. There's a good reason not to. The difference between Poisson and symplectic (at least in finite dimensions) is that a Poisson thing is locally a family of symplectic things (and globally maybe more complicated), which can jump in dimension. So as soon as you want to have "parameters" in your physics (say, the boundary value of a field appearing in a path integral), then you don't want symplectic: you want Poisson. And especially when you want "symmetries," including "non-manifest symmetries" and "dualities" and all the things that make physicisits very excited, you can't even work with Poisson things that are globally families of symplectics.

But, as I was saying, the structure of a differential which is a derivation mod $\hbar$, but actually second order — that structure is not a "BV algebra," because Getzler gave that name to a closely related but different thing having to do with Deligne's conjecture (now Theorem, probably Kontsevich and/or Tamarkin?) about Hochschild cohomology. Really Beilinson and Drinfel'd (in their big book on chiral algebras) were the ones to emphasize to mathematicians the importance of these structures, so Costello–Gwilliam propose the name "BD."

In nature, however, it turns out that the axioms of BD algebra are a little too strict. Here's a relaxation:

Definition: A *semistrict homotopy BD algebra* is a graded commutative algebra A along with a differential ∂ on $A[\![\hbar]\!]$, such that modulo $\hbar^N$, ∂ is an Nth-order differential operator. I also demand that $\partial(1) = 0$. So for example, it makes A itself into a dg algebra.

Recall that an *Nth-order differential operator*, according to Grothendieck, is a linear operator D such that for every element $f \in A$, the operator $g \mapsto D(fg) - fD(g)$ is an $(N-1)$th-order differential operator.

The word "semistrict" in the definition is the reminder that I'm not relaxing the commutativity or associativity of A, nor the Grothendieck definition of differential operator. I can get away with this if I'm in characteristic 0. In characteristic p, or over $\mathbb{Z}$, you really need to work with an even more homotopical definition.

The *principal symbol* of an Nth-order differential operator D is the derivation in N variables that measures the failure of D to be an $(N-1)$th-order differential operator.

Exercise: Let A be a semistrict homotopy BD algebra. Let L_N be the derivation in N variables such that $\hbar^{N-1}L_N$ is the principal symbol of ∂ on $A[\![\hbar]\!]/(\hbar^N)$. Then the L_Ns together make $A[-1]$ into an L_∞ algebra. What's an L_∞ algebra? It's a generalization of dg Lie algebra. It has a differential L_1. It has a bracket L_2, for which L_1 is a derivation. The Jacobi identity fails, but it's failure is the L_1-derivative of something L_3. And this L_3 satisfies its own version of Jacobi, but again only up to a homotopy L_4. Etc. If you want, the exercise is to unpack the definition of L_∞ algebra, to make the exercise true.

This justifies:

Definition: A *semistrict homotopy* Pois_d *algebra* is a graded commutative algebra A along with a sequence $L_1, L_2, \dots$ where L_N is a derivation in N variable, that together make $A[d-1]$ into an L_∞ algebra. ("Semistrict" again means that commutativity, associativity, and Leibniz rules are kept strict.)

Why the indexing? Well, L_N itself has degree $-1 + d(N-1)$, and when d is even, they're all symmetric in all of their inputs, and a few other reasons.

I should also mention: there are two versions of L_∞, the *flat* kind and the *curved* kind. The curved kind also has an operator L_0. It corresponds to BD algebras for which $\partial(1) \neq 0$. I don't like them as much.

There's some cool geometry here, but let me return to diagrams, which have to do with completed symmetric algebras.

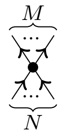
Definition: A *formal manifold* is a vector space of chain complex $X = \uparrow$, which we think of as the linear functions on the manifold, and we only will really care about the algebra $A = \widehat{\mathrm{Sym}}(X)$. Then you get things like *semistrict homotopy BD formal manifold* and *semistrict homotopy* Pois_d *formal manifold*, meaning such a structure on $\widehat{\mathrm{Sym}}(X)$, continuous for the power series topology. Note that any formal manifold has an *origin* where all $x \in X$ have value 0.

A structure is *coflat* if it only outputs things that vanish at the origin, i.e. $\partial(f) \in \langle X \rangle \subseteq A$ for all $f \in A$, where $\langle X \rangle$ is the closed ideal generated by X. For convenience, I'll only work with coflat things. Also, for organization purposes, I'd like to ask that $L_1 = \partial \mod \hbar$ have no linear part. I mean, it will have a linear part, but I'd like to keep it as part of the chain complex X. See, L_1 is a continuous derivation, so it's determined by its action on $X \subseteq A$, and if it's coflat, then $L_1(x) \in \langle X \rangle$ for all $x \in X$. So it's $L_1(x) = y + \dots$ where $y \in X$ depends linearly on x and $\dots \in \langle X^2 \rangle$ is quadratic or higher. Then the map $\partial_X : x \mapsto y$ is itself a dg structure, and makes X into a chain complex. So let's think of the data as $\big((X, \partial_X), \text{all the nonlinear stuff}\big)$.

Now we get to apply the homological perturbation lemma:

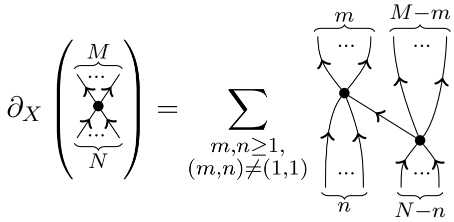
Corollary: In a (flat and coflat) semistrict homotopy BD formal manifold, there exists a differential $\tilde{\delta} = o(1)$ on $\widehat{\mathrm{Sym}}(\mathrm{H}_\bullet(X))[\![\hbar]\!]$ making that smaller algebra quasiisomorphic to $\big(\widehat{\mathrm{Sym}}(X)[\![\hbar]\!], \partial\big)$.
Proof: Extend ∂_X as a derivation. Then $\partial - \partial_X = o(1)$, and so apply HPL. (Actually, there's a more general proof, which shows that $\mathrm{H}_\bullet(X, \partial_X)$ is in fact itself a semistrict homotopy BD manifold.) **Corollary of the corollary:** In particular, if $\mathrm{H}_\bullet(X)$ is concentrated in degree 0, then $\mathrm{H}_\bullet\big(\widehat{\mathrm{Sym}}(X)[\![\hbar]\!], \partial\big) \cong \widehat{\mathrm{Sym}}(\mathrm{H}_0(X))[\![\hbar]\!]$ as vector spaces.

Now, the diagrammatics. The diagrammatics for semistrict homotopy Pois_d formal manifolds are a little easier. Each L_N is a derivation of N variables, so it's uniquely determined by a map $X^{\otimes N} \to A$. Expand it in Taylor series. You get a bunch of maps $X^{\otimes N} \to X^{\otimes M}$, and by (co)flatness $N, M \geq 1$, and by convention there's no map with $(N, M) = (1, 1)$. So in the diagrammatics we have a *corolla* (i.e. vertex) with N inputs and M outputs for each $N, M \neq 0$ and $(N, M) \neq (1, 1)$:



The outgoing strands transform trivially under permutations, and the incoming strands transform trivially if d is even or by the sign representation if d is odd.

These satisfy an equation of the form:

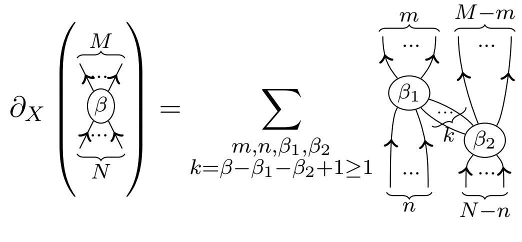


and you should sum or average over the ways to choose an output of the lower vertex to connect to an input of the outer one, and you should sum or average over ways to permute the incoming and outgoing strands, and you should perhaps include some numerical factors depending on your conventions for Taylor coefficients.

What about hBD? We can expand the differential ∂ on A in powers of $\hbar$: $\partial = \partial_1 + \hbar\partial_2 + \dots$, where ∂_K is an Kth-order differential operator on A. In coordinates, each ∂_K itself is a sum of terms, $\partial_K = \partial_{0,K} + \partial_{1,K} + \dots$, where $\partial_{N,K}$ is homogeneous kth-order differential operator, i.e. the canonical differential operator with its principal symbol. It's convenient to instead record the degree N and also $\beta = K - N$, which tells you how many extra powers of $\hbar$ there are that you didn't need for an operator of that order. Anyway, you get a bunch of corollas, after expanding the output in Taylor series:

with the rules that $M, N \neq 0$, and that $(M, N, \beta) \neq (1, 1, 0)$.

And again, there's a quadratic relation:



What's the diagrammatic interpretation of β? Well, it's the *genus* of the corolla. By definition, the *genus* of a connected graph is its first Betti number (i.e. $1-$ euler characteristic). If you declare that any corolla labeled β contributes an "internal genus" of β, then the defining equation above is homogeneous for genus.

Both of these are examples of *properads*:

Definition: An *associative algebra* is something with multiplications controlled by beads on a directed line. A *properad* is something with multiplications parameterized by connected directed acylic graphs (i.e. no oriented cycles, but genus is OK).

The variations are endless. You've probably heard of *operads*, which have multiplications controlled by rooted trees. Here's a few more. A *wheeled properad* uses connected directed graphs, that might have cycles. A *dioperad* uses directed trees. A *prop* uses possibly-disconnected directed graphs. I tend to work with *genus-graded properads*, for which in any equation the genus of the graph must be preserved, but vertices are allowed to have internal genus.

An *action* of any of these is an interpretation, compatible with the diagrammatics, in which the edge is assigned a chain complex, and the vertices/corollas/beads are assigned tensors.

The examples so far have been *quasifree*, meaning that the properad in question is free if you forget the dg structure, but has some interesting differential. Quasifree things are important because it's relatively easy to map out of them, and in particular to make them act. Especially if the generators are well-ordered and the differential of any generator is a composition of earlier generators. Then you can try to construct maps out of such a guy inductively. **Exercise:** By induction, the *obstruction* to defining a generator, namely the value of the right-hand side under and putative map, is closed. **Corollary:** You can continue the induction iff it vanishes in homology.

Here's some vocabulary from model category theory. Basically, "cofibrant" means "easy to map out of." "Fibrant" means "easy to map into." Slightly more precisely, for pretty much any category of dg algebraic gadgetry, you can declare that a map is a *fibration* if it is a surjection (and an object is *fibrant* if the canonical map to the terminal object is a fibration), and a *weak equivalence* or *acyclic* if it is an isomorphism in homology. Then an object P is *cofibrant* if for any acyclic fibration $Q \to R$, any map $P \to R$ admits a lift $P \to Q$. The basic theorem (in characteristic 0, for the way I've said it — you need that the representation theory of the symmetric group is semisimple, or you need to require that the generators only transform in projective representations of the symmetric group under permuting the incoming/outgoing strands) is that quasifree things with this well-ordering of the generators are always cofibrant.

Exercise: Prove this.

3 Poisson AKSZ

So far, we've talked about semistrict homotopy BD algebras and formal manifolds, and the diagrammatics and the homological perturbation theory thereof. The motivation is that these give generalizations of oscillating integrals. Implicit is that perhaps we will need such generalizations in order to formalize quantum field theory, or in order to define and then say things about interesting examples. But I haven't really said anything about how to construct nontrivial hBD formal manifolds.

Then again, maybe you believe that "quantum" has something to do with integrals (sum over states/histories...), but I certainly haven't talked at all about "field theory." So as motivation for some more abstract nonsense, let's really look at field theory.

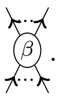
Let M be a smooth manifold, and X the vector space of linear functions on a formal manifold, so the formal manifold itself is X^*, at least if X is finite-dimensional. In *sigma models*, you define a *field* to be a map $M \to X^*$. So the space of all maps is $\mathscr{C}^\infty(M) \otimes X^*$, and this is supposed to be an infinite-dimensional manifold over which you can compute integrals, or impose equations of motion, or .... Let's treat it as a formal manifold — what's the vector space of linear functions on it? Why, it's $\mathscr{C}^\infty(M)^* \otimes X$, of course! What's $\mathscr{C}^\infty(M)^*$? It's the space of compactly-supported distributions on M.

Let's look instead at a particularly easy classical field theory. By definition, *classical field theory* is the study of PDE, or perhaps the part of PDE that has some Lagrangian or calculus-of-variations description. I'd like to focus on the easiest PDE, namely that of being constant. So my *classical equations of motion* for a map $\phi : M \to X^*$ are $\mathrm{d}\phi = 0$.

What's the space of all fields satisfying the classical EOM? It looks like X^*, or maybe a product thereof for the different components of M. Now, physics only has "local" measurements, so we should work only with (co)sheaves. And since we're already doing some homotopical stuff, probably we should work only with homotopy sheaves. The homotopy sheafification of the presheaf that assigns X^* to every open is exactly $\Omega_{\mathrm{dR}}(-) \otimes X^*$, with the de Rham differential. So the *derived* space of solutions to the EOM is $\Omega_{\mathrm{dR}}(M) \otimes X^*$, and so the vector space of linear functions on this is $\mathrm{Chains}_\bullet(M) \otimes X$, with the boundary map ∂_{dR}. Really, $\mathrm{Chains}_\bullet$ could be any homotopy cosheaf computing homology. A good one on smooth manifolds is $\mathrm{Chains}_\bullet = \Omega_{\mathrm{cpt}}$, the complex of compactly-supported de Rham forms. This is a formal manifold, and so the algebra of functions is $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(M) \otimes X)$.

What if you could give this infinite-dimensional formal manifold an hBD structure? Rewinding the logic from getting an hBD structure out of an oscillating integral, you'll see that an hBD structure on the formal manifold $\mathrm{Chains}_\bullet(M) \otimes X$ should be interpreted as a way of integrating against an oscillating measure over $\mathscr{C}^\infty(M) \otimes X$, where the "action" s imposes the classical EOM as its critical locus.

Let's be more down to earth, and set $M = \mathbb{R}$. Then we're doing "topological quantum mechanics in X^*" — "quantum mechanics" because we're on $\mathbb{R}$, and "topological" because the EOM are $\mathrm{d}\phi = 0$, so time durations don't matter. (It's topological in $\mathbb{R}$, not in X^*. The "topological" in "topological quantum field theory" is about the source manifold M, not the target.)

Look at each corolla . It corresponds to a map $\mathrm{Chains}_\bullet(\mathbb{R})^{\otimes \#\mathrm{inputs}} \otimes X^{\otimes \#\mathrm{inputs}} \to \mathrm{Chains}_\bullet(\mathbb{R})^{\otimes \#\mathrm{outputs}} \otimes X^{\otimes \#\mathrm{outputs}}$, or equivalently a tensor $X^{\otimes \#\mathrm{inputs}} \to X^{\otimes \#\mathrm{outputs}}$ where each entry is valued in the space of linear maps $\mathrm{Chains}_\bullet(\mathbb{R})^{\otimes \#\mathrm{inputs}} \to \mathrm{Chains}_\bullet(\mathbb{R})^{\otimes \#\mathrm{outputs}}$.

In physical situations, you should expect that such a linear map is sufficiently continuous to have an integral kernel, which would then be a distribution in $\mathbb{R}^{\#\mathrm{inputs} + \#\mathrm{outputs}}$.

Definition: An hBD structure on the formal manifold $\mathrm{Chains}_\bullet(\mathbb{R}) \otimes X$ is *quasilocal* if for each corolla, the corresponding integral kernel is supported in a finite-radius neighborhood of the diagonal $\mathbb{R} \hookrightarrow \mathbb{R}^{\#\text{inputs}+\#\text{outputs}}$.

In quantum field theory, you talk about "placing" observables at locations (or perhaps smeared out near locations) in spacetime. A quasilocal corolla can move observables, but not very far, and nearby observables can interact, but far away ones cannot. That's the idea, anyway. This is what happens for example in lattice field theory. Lattice field theory isn't strictly *local*, because e.g. neighboring sites can interact. But far away ones don't, or do only through nearby things.

Set $\mathcal{A} = \widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}) \otimes X)$ and $A = \widehat{\mathrm{Sym}}(X)$, and give $\mathcal{A}[\![\hbar]\!]$ the differential ∂ coming from an hBD structure. Modulo $\hbar$, this is just ∂_{dR}, and so $\mathrm{H}_\bullet(\mathcal{A}) = A$. Indeed, we have inclusion maps for each bump function, and a canonical projection (integration of chains). Now turn on the differential: by the HPL, at least if X is concentrated in degree 0, we still have $\mathrm{H}_\bullet(\mathcal{A}[\![\hbar]\!], \partial) \cong A[\![\hbar]\!]$. Remember, the map $\tilde{p} : \mathcal{A}[\![\hbar]\!] \to A[\![\hbar]\!]$ deforming the projection map is a type of "expectation value."

Now, let's take two observables f_1 and $f_2 \in A$. Place one at z_1 and one at $z_2 \in \mathbb{R}$, with $z_1 < z_2$. Multiply them in $\mathcal{A}[\![\hbar]\!]$ — this is not a chain map, since ∂ is not a differential — and then apply $\tilde{p}$ to get back to A. This is the *two point function*.

Lemma: Modulo large powers of X and $\hbar$, the two-point function is independent of z_1 and z_2 provided $z_2 \gg z_1$. **Proof:** Modulo large powers of X and $\hbar$, only finitely many corollas participate in ∂. These corollas all together are supported in some finite-radius neighborhood of a diagonal. So the failure of ∂ to be a derivation is supported only when z_1 and z_2 are near each other. HPL implies that if you vary z_2 in the absence of z_1, then you get a homologous inclusion map. In the presence of z_1, it's still homologous, exactly because ∂ is acting like a derivation. And the two-point function begins and ends with $A[\![\hbar]\!]$, which is concentrated in degree 0, and so homologous things are equal.

Theorem: The two-point function defines an associative multiplication on $A[\![\hbar]\!]$. **Proof:** Draw a picture. The point is that the "spacetime" $\mathbb{R}$ is precisely the string along which beads in an associative algebra are strung.

So this leads us back to abstract nonsense of properads, with the following question:

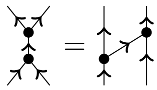
Question: Can you give $V \otimes W$ the structure of a formal hBD manifold by putting structures on V and W separately? Can such structures be put on $\mathrm{Chains}_\bullet(M)$?

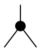
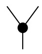
Answer: Yes. I will ignore various issues of finite-dimensionality, augmentations, and the like, and give the gist of it.

Let P be a genus-graded properad, and let $P(N, M, \beta)$ denote the chain complex of all operations with N inputs, M outputs, and total genus β. Its *bar dual* $\mathbb{D}P$ is a quasifree properad, defined as follows. The generators with N inputs and M outputs and internal genus β is the complex $P(N, M, \beta)^*[-1]$, the dual complex shifted down by 1 in homological degree. The differential encodes the binary composition in P. In particular, consider all graphs with two vertices. Each one defines a map $P \otimes P \to P$. Sum over all of them (best is to weight by number of automorphisms) to get another map $P \otimes P \to P$. The dual is a map $P^* \to P^* \otimes P^*$, and you can shift the homological degrees to get a degree-(-1) map $P^*[-1] \to P^*[-1] \otimes P^*[-1]$. This is the differential on generators of $\mathbb{D}P$.

I'll give examples in a moment, but first, the answer to the question. For any (...) genus-graded properad P, the map taking the corolla with N inputs, M outputs, and internal genus β to the canonical element in $P \otimes P^*[-1] \subseteq P \otimes \mathbb{D}P^*[-1]$ is a homomorphism. In particular, if V has an action by P, and W has an action by $\mathbb{D}P$, then $V \otimes W$ has an hBD structure.

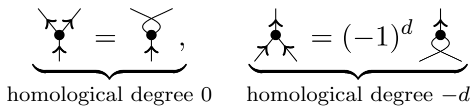
Example: An *open and coopen commutative frobenius algebra* is a vector space V which is a non-unital commutative algebra and a non-counital cocommutative algebra, satisfying the following identity:



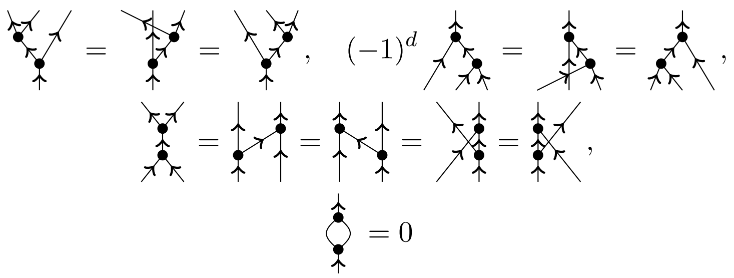
Here  is the multiplication and  is the comultiplication. These generate a properad Frob satisfying $\mathrm{Frob}(N, M, \beta) = \mathbb{K}$ (the ground ring of characteristic 0) unless N or $M = 0$, in which case the complex is 0.

The dual is $\mathbb{D}\mathrm{Frob} = \mathrm{hBD}$ itself. **Exercise.**

Example: The properad $\mathrm{invFrob}_d$ controlling *d-shifted involutive open and coopen commutative frobenius algebras* is similar. It has generators



and relations



Thus it satisfies $\mathrm{invFrob}_d(N, M, \beta) = 0$ if $\beta \geq 1$, or if N or $M = 0$. If $N, M \geq 1$, then $\mathrm{invFrob}_d(N, M, 0) = -d(N-1)$.

The dual is $\mathbb{D}\,\mathrm{invFrob}_d = \mathrm{hPois}_d$. **Exercise**.

There is always an acyclic fibration $\mathbb{D}\mathbb{D}P \to P$, meaning that $\mathbb{D}\mathbb{D}P$ is always a *cofibrant replacement* of P. Remember that cofibrant things are easier to act, and in particular play better with homotopical settings. A *homotopy action of P* is an action of $\mathbb{D}\mathbb{D}P$, or equivalently of any other cofibrant replacement.

Suppose that there was a *homotopy* action of $\mathrm{invFrob}_1$ — quasilocally — on $\mathrm{Chains}_\bullet(\mathbb{R})$. This isn't much to ask. For topological space M, $\mathrm{H}_\bullet(M)$ is a cocommutative coalgebra, and $\mathrm{H}^\bullet(M)$ is a commutative algebra. When M is an oriented d-dimensional manifold, we can embed $\mathrm{Chains}_\bullet \hookrightarrow \mathrm{Cochains}^{d-\bullet}$, and there are various (co)actions, and in homology there's a Frobenius-type axiom. Also, the involutivity is automatic for odd d. (For closed manifolds, it measures the Euler characteristic.) So I expected $\mathrm{Chains}_\bullet(\mathbb{R})$ to have a canonical homotopy $\mathrm{invFrob}_1$ action.

Anyway, suppose it did. Then we'd automatically get a hBD structure on $\text{Chains}_\bullet(\mathbb{R}) \otimes X$ whenever X is a hPois_1 formal manifold, and in particular whenever X is the linear functions for a coordinate chart on an honest Poisson manifold. This would give a canonical universal deformation quantization! Unfortunately, these don't exist.

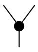
More generally, for those of you who know what this means... and E_d algebra is related to $\mathbb{R}^d$ the way associative algebras are related to $\mathbb{R}$. In particular:

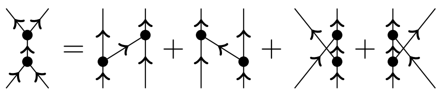
Theorem: Any quasilocal hBD structure on the formal manifold $\text{Chains}_\bullet(\mathbb{R}^d) \otimes X$ determines an E_d algebra structure on $\widehat{\text{Sym}}(X)[\![\hbar]\!]$. The principal symbols of determine a hPois_d structure.

Corollary: Any quasilocal homotopy invFrob_d action on $\text{Chains}_\bullet(\mathbb{R}^d)$ gives a universal quantization procedure for E_d. Because of known (hard!) calculations of $\text{H}_\bullet(E_d)$, this is equivalent to the formality of the E_d operad. **Conjecture:** The space of quasilocal homotopy invFrob_d actions on $\text{Chains}_\bullet(\mathbb{R}^d)$ is canonically equivalent to the space of formality morphisms of E_d. Note that this space is known to be non-empty, by deep work of Kontsevich.

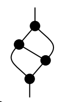
Back to dimension 1, it turns out that there is an obstruction to putting a quasilocal homotopy invFrob_1 action on $\text{Chains}_\bullet(\mathbb{R})$. But you almost can:

You know what a Lie bialgebra is? It's a Lie algebra, and a co Lie algebra, and a relation. I want a degree-shifted version:

Definition: The properad LB_1 is generated by a degree-0 *bracket*  satisfying the Jacobi identity, and a degree-(-1) *cobracket*  satisfying a signed coJacobi identity, and these together satisfy:



Theorem: There are cofibrant replacements $\mathbb{D}\,\text{LB}_1 \to \text{invFrob}_1$ and $\text{hPois}_1 = \mathbb{D}\,\text{invFrob}_1 \to \text{LB}_1$. **Proof:** Say a bunch of magic words about "Koszul duality."

Definition: I call by *surinvolutivity* (a souped-up version of involutivity) the relation  $= 0$.

Modding out LB_1 by the ideal generated by this relation gives the properad surinvLB_1.

Theorem: $\mathbb{D}\,\text{surinvLB}_1$ does act quasilocally on $\text{Chains}_\bullet(\mathbb{R})$.

Corollary: The cofibrant replacement $\mathbb{D}\mathbb{D}\,\text{surinvLB}_1$ is a quotient of $\mathbb{D}\mathbb{D}\,\text{LB}_1$, which is quasiisomorphic to hPois_1. Thus a *surinvolutive Poisson formal manifold* is a Poisson formal manifold whose $\mathbb{D}\mathbb{D}\,\text{LB}_1$ action restricts to $\mathbb{D}\mathbb{D}\,\text{surinvLB}_1$. There is a canonical (wheel free!) deformation quantization of surinvolutive Poisson formal manifolds. By "wheel free" I mean that the manifold can be infinite-dimensional — Kontsevich's universal quantization of arbitrary Poisson formal manifolds requires traces.

Closing question: Have you seen this relation on a Lie bialgebra before? What's a better name than "surinvolutive"?