Lie bialgebra quantization via 2- and 3-dimensional field theory

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Drinfeld Associator Seminar, Northwestern University.

This talk is based on — what's the phrase? — joint work on haitus with Owen Gwilliam: we had a series of conversations a few years ago but never wrote anything. So I should also say: thank you for the invitation. In addition to the pleasure of speaking, perhaps this will give me the motivation to write up these ideas.

My goal for the talk is to describe a certain "field theoretic" relationship between two proofs — by Etingof + Kazhdan and by Tamarkin — of the existence of universal/functorial quantization of Lie bialgebras. In a little more detail, there is a space (the main topic, I gather, of this seminar), which Drinfeld proved to be inhabited (= non-empty) using differential equations + homological algebra:

$$\begin{tikzcd} \{\text{Drinfeld associators}\} \ni \text{elements constructed by Drinfeld} \arrow[d, "\text{and EK constructed}"'] \\ \{\text{Lie bialgebra quantization functors}\} \end{tikzcd}$$

Their proof is basically a detailed study of "highest weight theory"/"Verma modules" for doubles of Lie bialgebras. In particular, it feels very representation-theoretic.

What Dima did was to factor their map:

$$\begin{tikzcd}
\{\text{associators}\} \arrow[rr, "EK"] \arrow[dr, "T"] & & \{\text{quantization functors}\} \arrow[dl, leftrightarrow, "T"] \\
& \left\{\begin{matrix}E_2 \text{ formality} \\ \text{morphisms}\end{matrix}\right\} &
\end{tikzcd}$$

His proof is almost entirely homotopy theory of operads.

My goal will be to explain how these two proofs have interpretations within the same field theory. So if "statements" are 0-cells and "proofs" are 1-cells in some space, I will try to describe a 2-cell.

OK, so I'm going to put aside for a while associators and formality and so on, and talk about field theory. By definition, classical field theory is the study of PDEs. PDEs are too hard for me, but there is one PDE I basically understand: $df=0$. Solutions are locally constant functions.

Where should these functions take values?

I will work with <u>infinitesimal</u> spaces, which are by definition cocommutative coalgebras. ~~B~~A fundamental theorem of Sweedler's says that in char=0 every cocom. coalg. is a "disjoint union ~~B~~ of fuzzy points". Here is an example. Let V be a vector space. Then $\mathrm{Sym}(V)$ is a coalgebra, which you should think of as the infinitesimal nbhd of $0 \in V$. (My whole story is in char$=0$.) Actually, I will work ~~a~~ with dg cocom coalgs = "dg infinitesimal spaces". The category of (dg) cocom coalgs is cartesian closed.

For example, what are the <u>groups</u> in this category? ~~Let's ask just for the connected ones, ie as an infinitesimal space, they should have a single closed point.~~ So I want a cocom coalg H with a coalg. map $m: H \otimes H \to H$ (since $\otimes = \times$ in the category) which is associative, and I want an "inverse" $S: H \to H$ such that $m \circ (\mathrm{id} \otimes S) \circ \Delta = e \circ (\text{project to pt})$. These are precisely cocom Hopf algebras. Focussing just on the ones with a single closed point ~~for~~ (the identity) gives, again by Sweedler, the Hopf algs of the form $U\mathfrak{g}$ for $\mathfrak{g}$ a Lie alg. So $U\mathfrak{g}$ is a <u>group</u>.

Any group should have a <u>bar</u> construction, at least in dg / model category land. I will drop the "U", and write $\mathfrak{g}$ for the group. (Note: $U\mathfrak{g} \simeq \mathrm{Sym}\,\mathfrak{g}$ as a coalg; I will ~~a~~ similarly drop "Sym", and write V for $\mathrm{Sym}\,V$.) We have

$$\mathfrak{g} \circlearrowright pt \ (= \mathrm{Sym}\, 0), \quad \text{and}$$

$$B\mathfrak{g} := pt \underset{\mathfrak{g}\text{-sets}}{\times} pt = \mathfrak{g}\text{-coinvariants of } pt \times pt \text{ under diag. action.}$$

$$= pt/\mathfrak{g}.$$

This makes sense in dg cogebrés, and gives

$$B\mathfrak{g} = CE_\bullet(\mathfrak{g}) = (\mathrm{Sym}(\mathfrak{g}[1]) \text{ w/ } \partial_{CE} \leftrightarrow \text{bracket.})$$

Conversely, $\mathfrak{g} = \Omega_\bullet B\mathfrak{g} = pt \underset{B\mathfrak{g}}{\cap} pt = pt \underset{B\mathfrak{g}}{\times} pt$ where $pt \to B\mathfrak{g}$ is $\mathrm{Sym}(0 \hookrightarrow \mathfrak{g}[1])$.

Let X be a manifold. What are the "locally constant maps $X \to B\mathfrak{g}$"? Heuristically, $B\mathfrak{g}$ = "groupoid of $\mathfrak{g}$-torsors," so a l.c. map is "a $\mathfrak{g}$-torsor at every $x \in X$, with local trivializations," i.e. a "$\mathfrak{g}$-bundle with flat connection."

Let's do this more precisely. A locally constant map $X \to B\mathfrak{g}$ should assign a point in Bog for each pt $x \in X$, but any path in X should give an isomorphism of the values at the endpoints, etc., so we define

~~Maps~~ Locally constant Maps (X, Bog)

$$\parallel$$

$$\underline{\mathrm{Maps}}(C_\bullet X, Bog)$$

Here "$C_\bullet X$" is (any) chain model, with comultiplication $C_\bullet X \xrightarrow{\Delta} C_\bullet(X \times X) = C_\bullet X \otimes C_\bullet X$. (Depending on the model, you might have to work a little to make this a strict cocom coalg.) And "$\underline{\mathrm{Maps}}$" is the inner hom in the category of coalgebras, satisfying

$$\hom(A, \underline{\mathrm{maps}}(B, C)) = \hom(A \times B, C)$$

categorical $\times = \otimes$

Now you can compute:

$$\underline{\mathrm{Maps}}(A, (\mathrm{Sym}(V), \partial_V))$$

any cocom coalg.

a "quasi-free" coalg, i.e. $\mathrm{Sym}\, V$ as a graded thing, w/ some differential d determined by $\{\partial_V^{(n)} : \mathrm{Sym}^n V \to V\}_{n \geq 1}$

$$= (\mathrm{Sym}(A^* \otimes V), \partial_{\mathrm{Maps}})$$

the dual algebra

$$\partial^{(n)}_{\mathrm{Maps}} = m^{(n)} \otimes \partial^{(n)}_V$$

n-fold multiplication in A^*.

So in our case we get

$$\underline{\mathrm{Maps}}(C_\bullet X, Bg) = \underline{\mathrm{Maps}}(C_\bullet X, (\mathrm{Sym}(g[1]), \partial_{CE})$$

$$\| $$

$$\left(\mathrm{Sym}(C^\bullet X \otimes g[1]), \partial_{\underline{\mathrm{Maps}}} = \partial_{C^\bullet X} + \partial_{CE}\right)$$

$$\| $$

$$CE_\bullet(C^\bullet X \otimes g).$$

If X is a manifold, we can reasonably set $C^\bullet X = \Omega^\bullet_{dR} X$. If X is oriented, it is not unreasonable to set $C_\bullet X = \Omega^{\dim X - \bullet}_{cpt} X$, although you have to work a bit to make that into a coalgebra.

So much for classical field theory. What would quantizing entail?

According to Feynman, one source of quantum field theories should be when the classical field theory is cut out as the critical points of an "action" S.

Then the quantum theory should have to do with integrating against $\exp(\frac{i}{\hbar} S)$.

How can we detect if a given space of fields, e.g. $CE_\bullet(C^\bullet X \otimes g)$ $= \underline{\mathrm{Maps}}(C_\bullet X, Bg) = B(C^\bullet X \otimes g)$, is cut out as a critical locus? One way to detect this is that the derived critical locus of any function is

This is called "BV integration".

symplectic w/ $\deg(\omega) = +1$ (homological convention)
In this case, integrating against $\exp(\frac{i}{\hbar} S)$ is related to integrating a semidensity on this symplectic space (w/ prescribed $\hbar \to 0$ leading asymptotic) against a "gauge-fixing lagrangian" (whose role includes improving the convergence of the integral).

So we can ask: under what circumstances is

$$\underline{\mathrm{Maps}}(C_\bullet X, A)$$

symplectic w/ ~~deg~~ $|\omega| = d$?

Focussing on ($A = \mathrm{Sym}\, V, \partial_V$) the tangent bundle to $\underline{\mathrm{Maps}}(C_\bullet X, A)$ is modeled on $C^\bullet X \otimes V$, with differential a linearization of ~~the~~ ∂_V. (Compare $A = B\mathfrak{g}$; then $T_p \underline{\mathrm{Maps}}(C_\bullet X, B\mathfrak{g}) = C^\bullet X \otimes \mathfrak{g}[1]$ w/ differential coming from the connection on p, "ad p")

So another version of the question is: when does $C^\bullet X \otimes V$ have a nondegenerate pairing (varying "flatly" in the point $p \in \underline{\mathrm{Maps}}$)? Well, we would have such a pairing if V itself had a pairing and if X were compact and oriented. Then we could set

$$\langle \overset{\in C^\bullet}{f_1} \otimes \overset{\in V}{v_1}, f_2 \otimes v_2 \rangle = \left(\int f_1 \wedge f_2\right) \langle v_1, v_2 \rangle.$$

~~So~~ Tracking degrees, we see that

Thm [AKSZ]:

If $A=(\mathrm{Sym}\,V, \partial_V)$ is a symplectic cogebra w/ $|\omega| = m$, and X is an oriented compact manifold of dimension n, then

$$\underline{\mathrm{Maps}}(C_\bullet X, A) = \left(\mathrm{Sym}(C^\bullet X \otimes V), \partial_{dR} + \partial_V\right)$$

is symplectic w/ $|\Omega| = m+n$.

If X is not compact, then we can't integrate arbitrary forms. But we can foliate $T\underline{\mathrm{Maps}}$ by $C^\bullet_{cpt} X \otimes V$, the compactly supported forms, and this has a pairing. This foliation looks like the filtration by symplectic leaves in a Poisson manifold.

Poisson geometry is subtle in infinite dimensions. But ~~it~~ ~~appro~~ nevertheless:

Thm [me]:

If $A = (\mathrm{Sym}\,V, \partial_V)$ is homotopy Poisson w/ $|\pi| = m$, and X is oriented ~~, cpt~~ of dim n, then $\underline{\mathrm{Maps}}(C_\bullet X, A)$ is ~~Poisson~~ homotopy Poisson w/ $|\pi| = ~~m~~\, m-n$.

You can pose some of the "BV quantization" problem with Poisson rather than symplectic "classical" data. The cost is that you don't get numbers out, but more like "functions on the space of symplectic leaves." (I don't have a precise statement of this interpretation of the output.)

I started this story by promising something about Lie bialgebras. Let $\mathfrak{g}$ be a Lie bialgebra, i.e. a Manin triple $\mathfrak{g} \hookrightarrow \mathfrak{g} \bowtie \mathfrak{g}^* \hookleftarrow \mathfrak{g}^*$. Then:

<u>Example</u>: • $B\mathfrak{g}$ is P_2, i.e. Poisson w/ $|\pi| = 1$.

• $B(\mathfrak{g} \bowtie \mathfrak{g}^*)$ is symplectic w/ $|\omega^{-1}| = 2$. {More generally, if $\mathfrak{k}$ is a Lie alg w/ $\Theta \in (\mathrm{Sym}^2 \mathfrak{k})^{\mathfrak{k}}$, then $B\mathfrak{k}$ is P_3.} Here Θ^{-1} = pairing $\mathfrak{g}$ w/ $\mathfrak{g}^*$.

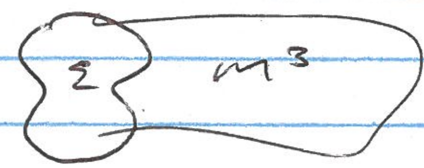
BV integrals want P_0, i.e. $|\pi| = -1$. So:

- If Σ^2 is an oriented surface, then $\mathrm{Maps}(C_\bullet \Sigma, B\mathfrak{g})$ is P_0.
- If M^3 is an oriented 3-manifold, then $\mathrm{Maps}(C_\bullet M, B(\mathfrak{g} \bowtie \mathfrak{g}^*))$ is P_0.

So we can pose quantization problems, and the latter has an "integration" interpretation $\leftrightarrow$ tools from thy of perturbative oscillatory integrals.

You can also describe field theories with boundary conditions. E.g:

$$\Sigma^2 = \partial M^3$$

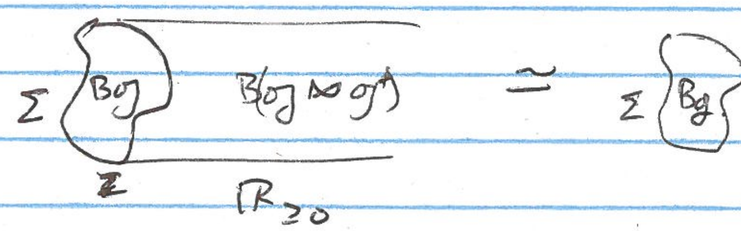


Look at

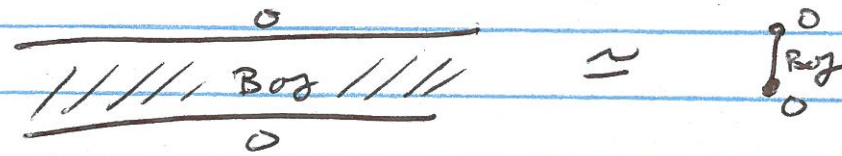
$$\underline{\mathrm{Maps}}(\Sigma, B\mathfrak{g}) \times_{\underline{\mathrm{Maps}}(\Sigma, B(\mathfrak{g}\ltimes\mathfrak{g}^*))} \underline{\mathrm{Maps}}(M, B(\mathfrak{g}\ltimes\mathfrak{g}^*))$$

Since $B\mathfrak{g} \hookrightarrow B(\mathfrak{g}\ltimes\mathfrak{g}^*)$ is Lagrangian, this is symp. if M is compact. Even if M were not compact, it's enough for b.c.s to be coisotropic for the whole thing to be Poisson.

Let's look at $M = \Sigma \times \mathbb{R}_{\geq 0}$. The data of a loc. const. map $M \to$ target is the same as its restriction to $\Sigma \times \{0\}$, since $\mathbb{R}_{\geq 0}$ is contractible. So



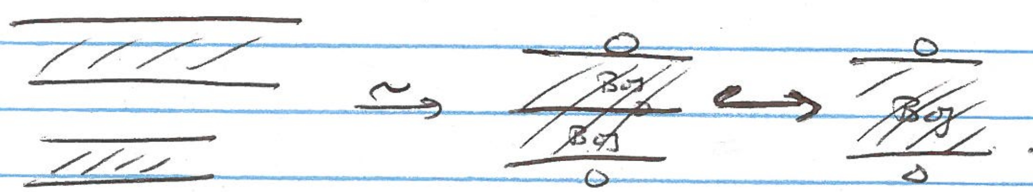
Here's another example. Look at a strip



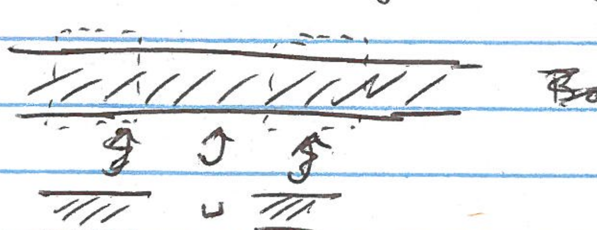
with b.c.s ~~pt~~ $\xrightarrow{0}$ $B\mathfrak{g}$ on both sides. (This is a poisson submanifold, and in particular coisotropic.) The space of loc. const. maps is the

$$pt \times_{B\mathfrak{g}} pt = U\mathfrak{g}.$$

The alg. structure is by fusing  i.e.



The comultiplication can be understood as restricting ~~maps~~ fields along



$$U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}.$$

If we just work with classical field theory, we see (constructible) sheaves, which can be restricted along arbitrary maps (or, say, open immersions).

<u>Defn</u>: A (constructible) <u>factorization coalgebra</u> is like a (constructible) sheaf but w/ restriction maps only along embeddings.

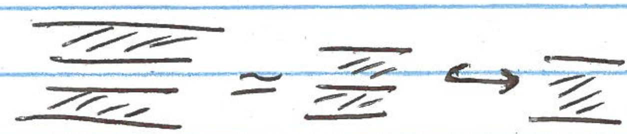
<u>Motivation</u>: These are the only restriction maps that we can guarantee to be Poisson in ~~this~~ our examples.

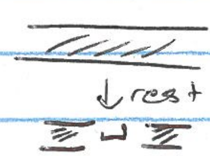
<u>Defn</u>: A <u>deformation quantization</u> of a classical field thy is a (perturbative) factorization coalg deforming in the P_0 direction.

<u>Lemma</u>: Deformations of constructible things are automatically constructible, since deformations cannot add homology.

<u>Lemma</u>: Constructible factorization coalgs on

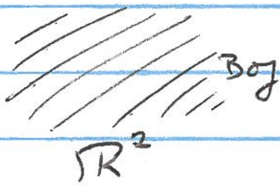
with ~~value~~ stalk 0 on the boundaries ~~that are preserved under deformation!~~ are Hopf algs.

m: 

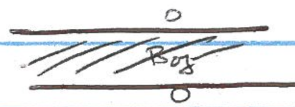
Δ: 

Dima's proof of Lie bialg quantization:

(1) E_2 formality gives a quantization of constr. fact. coalgs on $\mathbb{R}^2$.

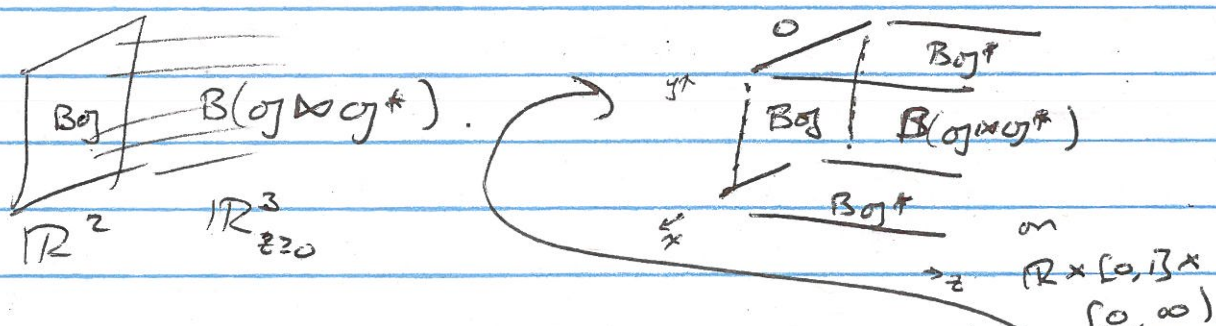


(2) Check behavior for augmented things to use this on



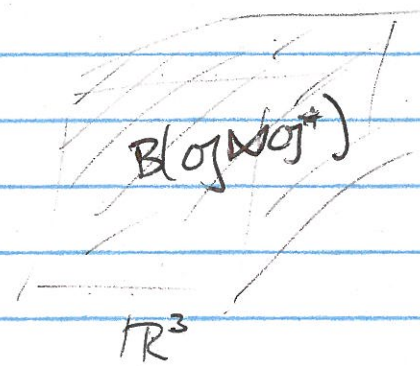
. ∎ QED.

This is great if you already have E_2 formality. What if you don't? Then, following Kontsevich, you might try to turn the problem into one where techniques from path integrals apply. i.e. make it more like a symplectic case. We said already how to do this: use 3-dim manifolds w/ target $B(\mathfrak{g}\bowtie\mathfrak{g}^*)$:



Actually, to really make it match the strip, we use

First let's just try to quantize in the bulk:



This is a "Chern-Simons" theory for $\mathfrak{g} \bowtie \mathfrak{g}^*$. The degree-0 observables are "Wilson lines" for representations of $U(\mathfrak{g} \bowtie \mathfrak{g}^*)$. There are two most important representations, which you should think of a "verma modules", one with "highest weight zero" and the other "lowest weight zero". They are:

$$U_- = \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{g} \bowtie \mathfrak{g}^*} \mathrm{triv} = U(\mathfrak{g} \bowtie \mathfrak{g}^*) \otimes_{U\mathfrak{g}} (\mathrm{triv}).$$

$$U_+ = U(\mathfrak{g} \bowtie \mathfrak{g}^*) \otimes_{U\mathfrak{g}^*} (\mathrm{triv}).$$

As sheaves on $B(\mathfrak{g} \bowtie \mathfrak{g}^*)$, these are the pushed-forward structure sheaves for ~~$B\mathfrak{g}$~~

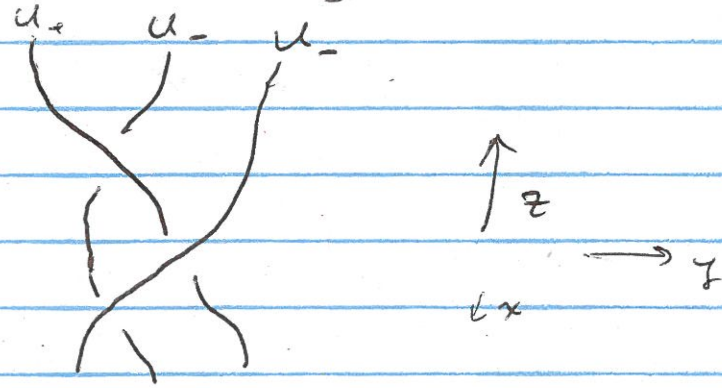
$B\mathfrak{g} \hookrightarrow B(\mathfrak{g} \bowtie \mathfrak{g}^*)$, and

$B\mathfrak{g}^* \hookrightarrow B(\mathfrak{g} \bowtie \mathfrak{g}^*)$.

In particular, $\mathrm{End}_{\mathfrak{g} \bowtie \mathfrak{g}^*} U_- = U\mathfrak{g}$, and similarly for U_+. Also PBW says

$$U_- \cong U\mathfrak{g}^* \text{ non-canonically.}$$

These modules are infinite-dimensional, so I should not ask for traces of holonomies. But I can ask for holonomies along, say, braids:



The Poisson structure tells you that:

$\{$  $-$  $\}$ $= \hbar \cdot$ action of Θ on $U_- \otimes U_+$ $+ O(\hbar^2)$

where $\Theta = \epsilon\begin{pmatrix} \mathfrak{g} \otimes \mathfrak{g}^* \\ \mathfrak{g}^* \otimes \mathfrak{g} \end{pmatrix} \in U(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes U(\mathfrak{g} \oplus \mathfrak{g}^*)$ is the dual to the pairing.

To implement this, we can use Feynman diagrams. The details depend on a gauge-fixing condition and quantum corrections to the measure. Changes to the gauge fixing shouldn't effect the output. There is a gauge-fixing condition on the Feynman diagrams which only has propagators running "horizontally" (constant z). Propagators are labeled by Θ, and their weights can be forced to depend only on the ~~x,y,z~~ direction. The end result is:

$$X = \exp(\tfrac{\hbar}{2}\theta) \qquad X^{-1} = \exp(-\tfrac{\hbar}{2}\theta)$$

 $= \Phi(\theta)$ for Φ a Drinfeld associator

and any Φ works. E.g. ~~if you~~ Kontsevich found a set of quantum corrections by demanding conformal invariance in the (x,y) plane, and I think this one corresponds to the AT associator, but I could be misremembering.

So we get a new braided $\otimes$ category from these Wilson lines. Classically,

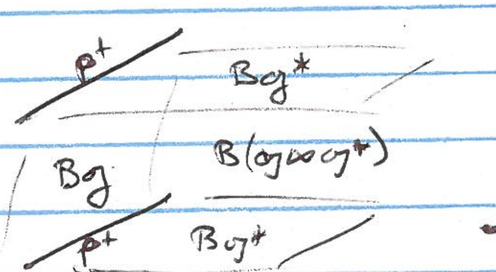
$$U\mathfrak{g} = \mathrm{End}_{\mathfrak{g}\bowtie\mathfrak{g}^*} U_- .$$

In the new category, we can try to do the same thing. To get a Hopf algebra requires some monoidality of the functor $\hom(U_-, -)$. Details are as in the Etingof-Kazhdan paper.

So what's the point? The point is that $U(\mathfrak{g}\bowtie\mathfrak{g}^*)$-mod has a subcategory corresponding ~~to $U(\mathfrak{g})$-mod~~ ~~gen~~ of things you get by inducing from $U\mathfrak{g}$-mod. ~~These~~ In terms of $B(\mathfrak{g}\bowtie\mathfrak{g}^*)$, these are the sheaves on $B\mathfrak{g} \hookrightarrow B(\mathfrak{g} \bowtie \mathfrak{g}^*)$. What EK did was to find a monoidal subcategory of the deformed category of $(\mathfrak{g} \bowtie \mathfrak{g}^*)$-modules defining this subcategory; Tannakian reconstruction then gives a (quasi...) Hopf algebra.

The full braided category, then, is the category of line defects in the bulk of our 3-d theory, with target $B(\mathfrak{g} \bowtie \mathfrak{g}^*)$. The subcategory consists of those defects that can end on the lagrangian boundary $B\mathfrak{g}$.

There's a little more work required to get the details precisely right. But this should be how the EK proof is related to (quantizing) the qft I described earlier.



Eventually, we'd like to be able to apply Costello–Gwilliam machinery to directly quantize these different theories, thereby giving a third proof of Lie bialgebra quantization. If we can, we hope the field theoretic proof really implements the Tannakian argument on the boundary and also EK in the bulk. But we're not there yet.