The Jones Polynomial and the Temperley–Lieb TQFT

Theo Johnson-Freyd, 9 May 2014
Northwestern Graduate Student Seminar.

0. Goal of the talk, and a joke.

My main goal for this talk is to introduce and then neatly package some famous objects from quantum topology. So I should begin by saying what that is:

Theorem: TFAE, and define the field of quantum topology:

(1) low-dimensional topology $\cap$ quantum algebra.

(2) that part of mathematics stemming from the Jones polynomial.

I won't try to define either part of (1), but I will give an ahistorical definition of the Jones polynomial. (Jones found it by thinking about lattice models of quantum statistical mechanics.)

I doubt I will say everything in these notes.

1. The Temperley–Lieb category

I will define a category $\mathcal{tl}$. Actually, I will give two definitions, with more to come; I hope it will be obvious that they give equivalent categories.

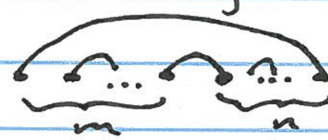
objects: $\mathrm{ob}(\mathcal{tl}) = \mathbb{N} = \{0, 1, 2, \dots\}$.

or

$$\mathrm{ob}(\mathcal{tl}) = \mathrm{Conf}^*(I) = \mathrm{Conf}^*([0,1])$$
= configurations of points in the interval.

The category tl will be <u>enriched</u> over $\mathbb{Z}_q := \mathbb{Z}[q, q^{-1}]$, meaning hom sets are $\mathbb{Z}_q$-modules and composition is linear in each variable.

<u>morphisms</u>: $\hom_{tl}(m, n)$ = the free $\mathbb{Z}_q$-module with basis the set of "non-crossing pairings" of $m+n$ things:

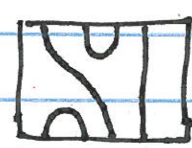


, but not

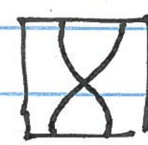
.

<u>Exercise</u>: there are none of these when $m+n$ is odd, and the $\left(\frac{m+n}{2}\right)$th Catalan number of them when $m+n$ is even.

$mor(tl)$ = ~~embeddings~~ or: free $\mathbb{Z}_q$-module with basis: embeddings of (unoriented, unlabeled) curves into $I \times I = [0,1]^2$ with end points on $I \times \{0,1\}$:



but not



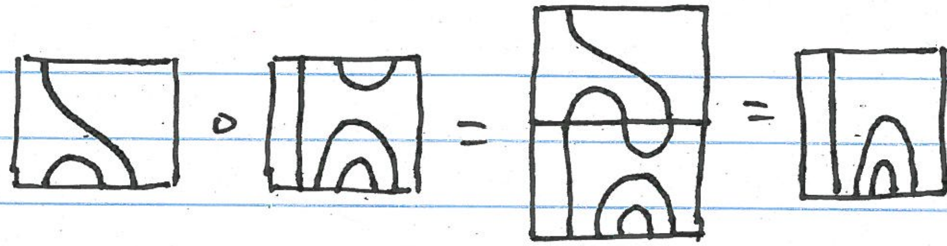
,

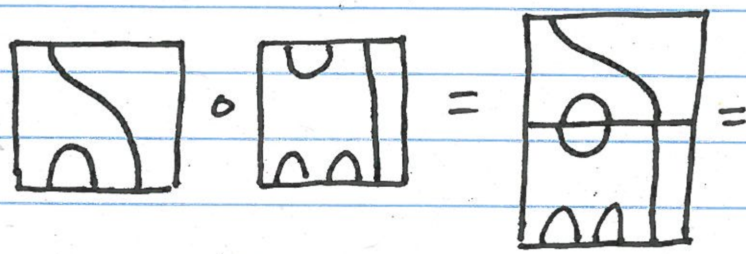
modulo ambient isotopy rel. boundary.

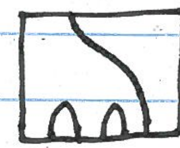
So far we haven't used the $\mathbb{Z}_q$-structure. It enters when we define composition. Composition can be defined purely combinatorially, but it's clearest in pictures.

<u>Composition</u>: "stack the boxes, and remove closed loops for a factor of $-q^2 - q^{-2}$."

Example:

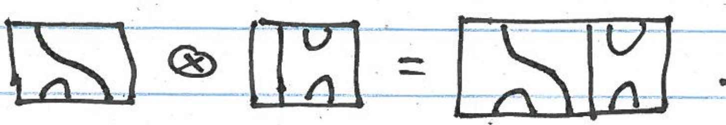




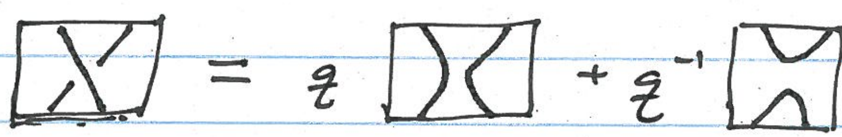
$= (-q^2 - q^{-2})$  .

This is the rule on basis vectors; you can extend by linearity.

tl is a monoidal category, meaning there is an associative bilinear functor $\otimes : tl \times tl \to tl$. In boxes it is "stack horizontally":

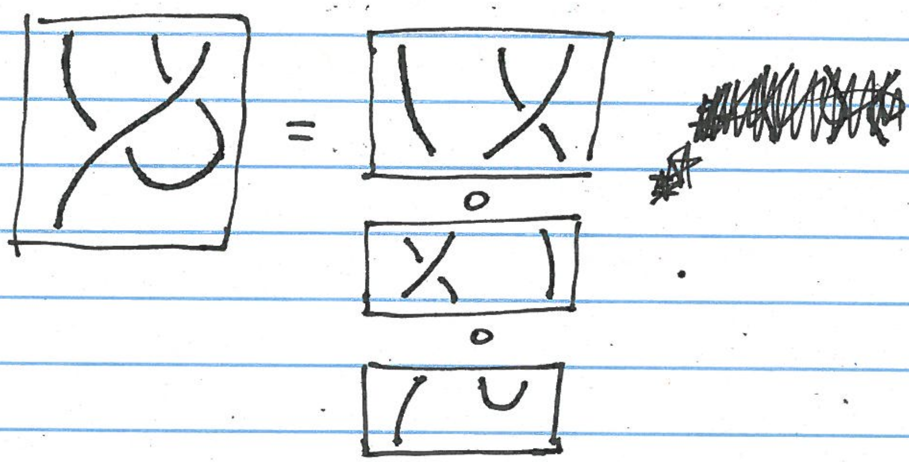


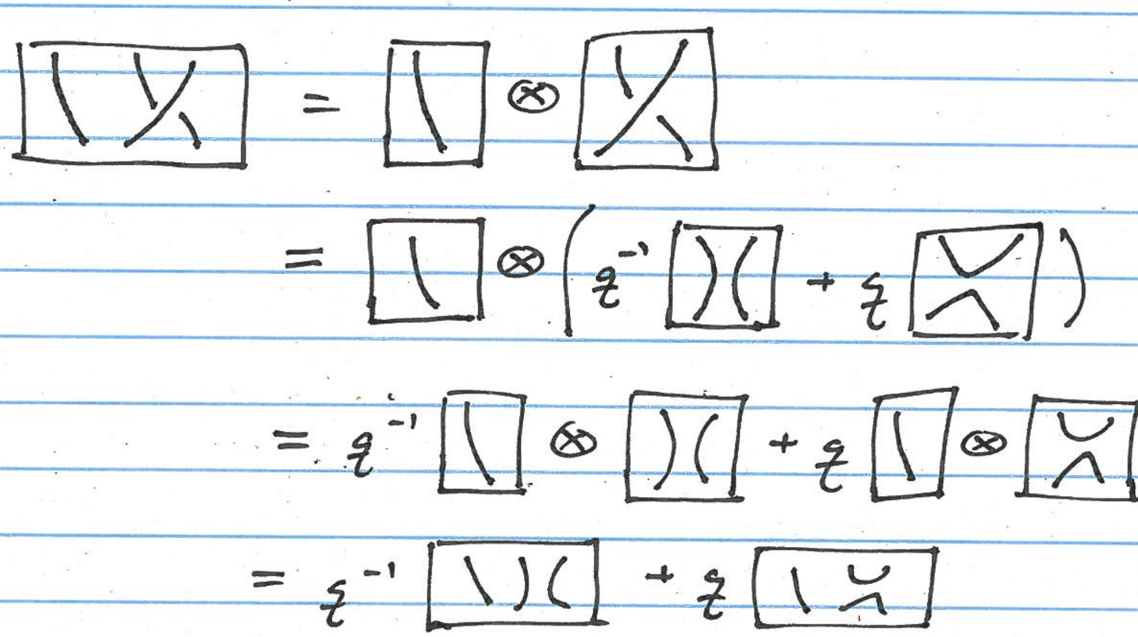
The reason for the qs (and addition) is the following. Let's define a new symbol:

 $\in \hom_{tl}(2,2)$.

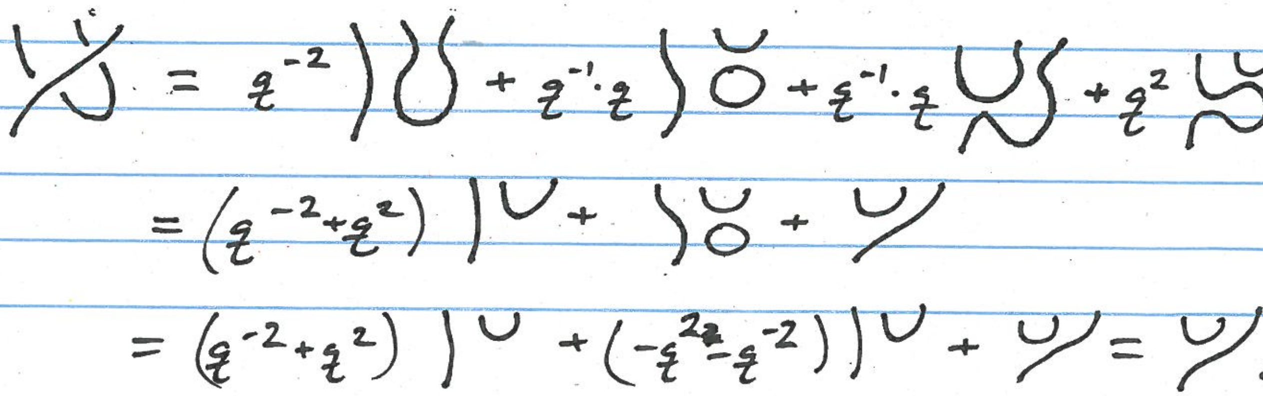
It's not a new element — it's already there. We also define

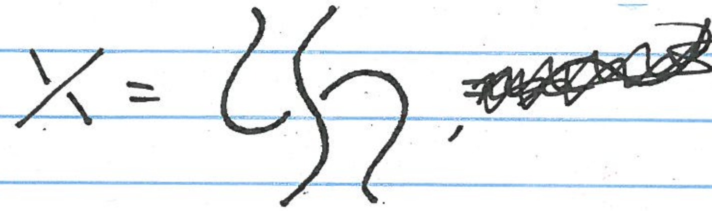
Then for any diagram with "crossings," we can expand it as a sum of basis vectors, using composition and tensor products:





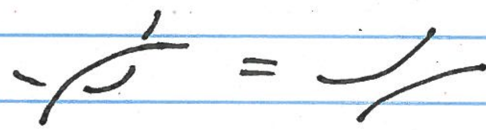
So, also distributing over composition,



<u>Exercise</u>: 

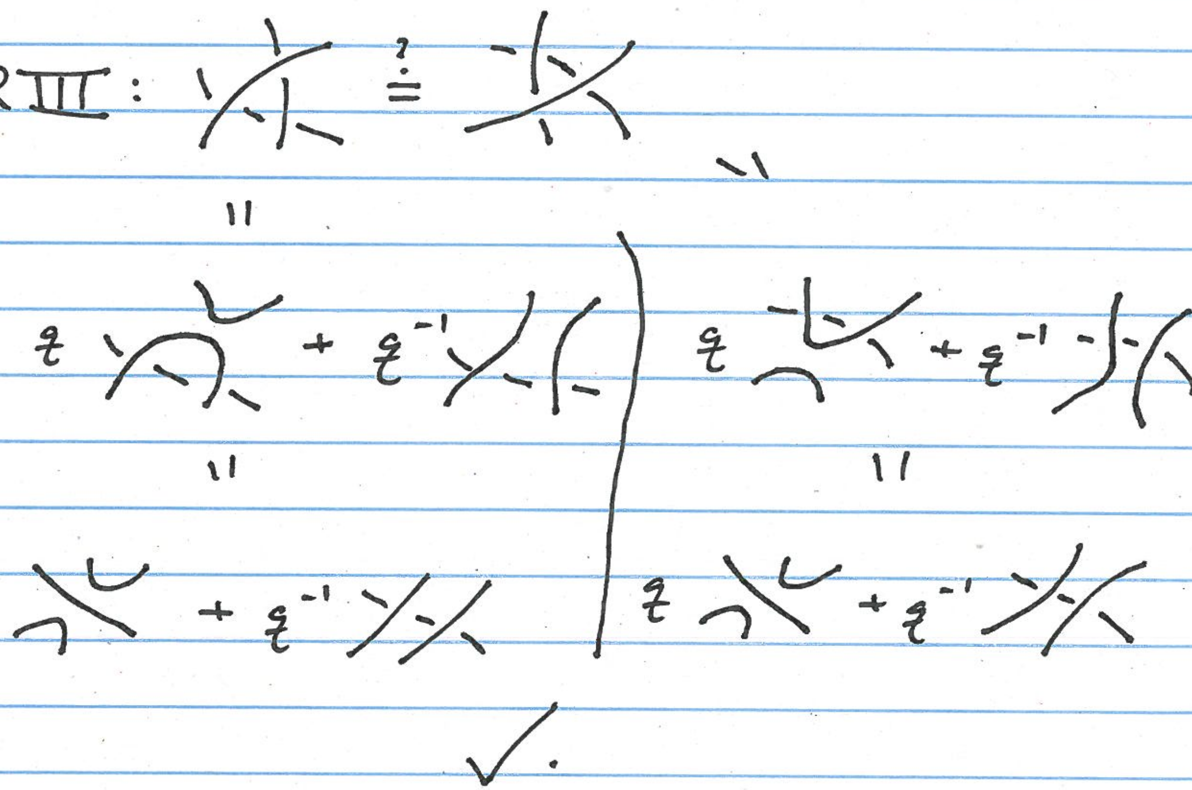
so the notation is consistent.

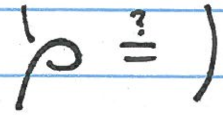
Then the previous calculation proves:

RII: 

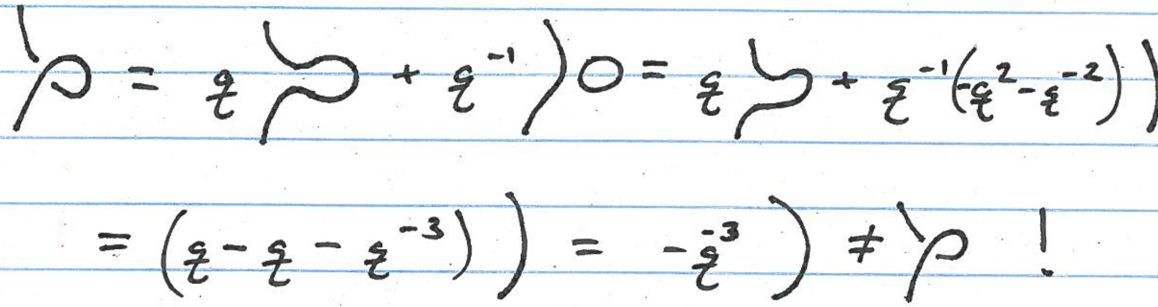
so  in $\hom_{TL}(2,2)$.

What else should we prove to justify the notation?

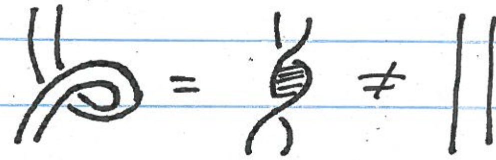
RIII: 

RI: 

Recall <u>Reidemeister's theorem</u>:
Any two ~~[illegible]~~ isotopic links are connected by a series of RI, RII, RIII moves.

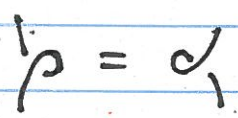


What's going on? We've introduced a "kink".
And indeed with a belt/ribbon,



but it does = .

<u>Theorem (Reidemeister)</u>: Any two isotopic <u>ribbon</u> links are related by RII, RIII, and

$R\frac{1}{2}$: .

<u>Exercise</u>: Check $R\frac{1}{2}$.

<u>Corollary</u>: Here's yet another category equivalent to $\mathcal{TL}$:

<u>Objects:</u> $\mathrm{Conf}^{fr}_*(D^2)$ = "framed" configurations of points in $D^2 = I \times I$ = .

"framed" means that we mark some points, and at each point choose a unit tangent vector.

$\mathbb{Z}_q$-span of:

<u>morphisms</u>: "ribbon tangles in $D^2 \times I$ ending at the marked points" / "skein relations".

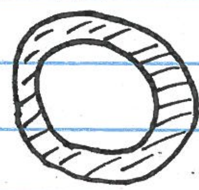
take these up to ambient isotopy rel boundary.

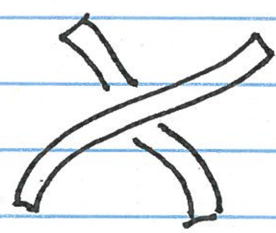
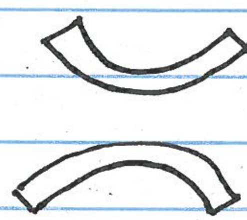
I will explain these with pictures:

$\in$ hom( , ).

The arrow tells you which side of the ribbon is satin, and which side is matte.

The relations are:

 $= -q^2 - q^{-2}$.

 $= q$  $+ q^{-1}$ 

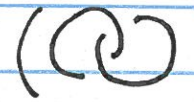
The functor giving the equivalence of categories is "project from $D^2 \times I$ to $I \times I$ (isotoping so that the projection is clean)." That this is well-defined follows from Reidemeister's theorem.

<u>Convention</u>: "blackboard" framing: in any diagram, the matte side is behind, shiny side up.

2. The Jones polynomial

Any ribbon knot (or link) defines an element of $\hom_{\mathcal{TL}}(\emptyset,\emptyset)$, if you use this "3-dimensional" description. But $\hom_{\mathcal{TL}}(\emptyset,\emptyset) = \mathbb{Z}_q = \mathbb{Z}[q,q^{-1}]$ by the "1-dimensional" description. The element defined by the knot K is its Jones polynomial (with "functorial normalization"), $J(K) \in \mathbb{Z}[q,q^{-1}]$.

E.g.

$J($  $) = q^2$  $+$  $+$  $+ q^{-2}$ 

$$= (q^2+q^{-2})\left((-q^2-q^{-2})^2\right) + -q^2-q^{-2}.$$

$\neq (-q^2-q^{-2})^2 = J($  $).$

Exercise: Compute $J($  $)$ and $J($  $)$.

So the Hopf link is not framed-isotopic to the unlink.

But in fact, remember that $|\!\!\rho = -q^{-3}\,|$.

So if two ribbon links are not framed-isotopic but still unframed-isotopic, then their Jones polynomials might not be equal, but will be equal up to a factor of $-q^{-3}$.

Exercise:  $\neq$  .

3. Packaging: a TQFT

I described three equivalent categories. Let me now call them by different names:

original $tl = tl(pt)$.

the one with boxes $= tl(I)$.

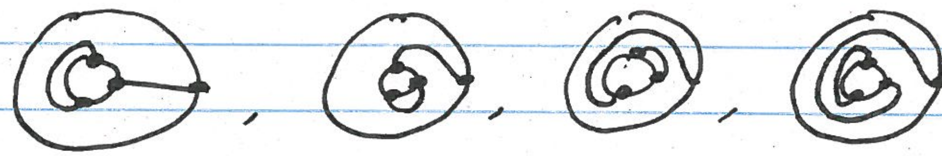
the one with a disk $= tl(D^2)$.

We can play the same game for other low-dimensional manifolds. E.g.:

$tl(S^1) =$ category with obj $= \mathrm{Conf}_*(S^1) \underset{\text{?}}{\cong} \mathbb{N}$.

morph $= \mathbb{Z}_q$-module with basis the non-crossing diagrams in $S^1 \times I$.

hom() $= \mathbb{Z}_q$-span of

{..., , ...}.

~~This is still a monoidal category~~

Maybe it's better to draw these as cylinders .

This is still a monoidal category by "stacking" in the transverse direction:

and resolving crossings using the skein relations.

If Σ is an oriented surface, we can define $tl(\Sigma)$ similarly: ~~ea~~ obj $= \mathrm{Conf}_*^{fr}(\Sigma)$ and mor $=$ ribbon tangles in $\Sigma \times I$. Now orientation matters in order to define the skein relation.

Note that in general $tl(\Sigma)$ is "just a category" — it doesn't have a monoidal structure. Actually, it does have some structure: it has a distinguished object $\emptyset_\Sigma \in tl(\Sigma)$, the configuration of no marked points.

<u>Defn</u>: Let Σ be a closed oriented surface. The <u>skein algebra</u> of Σ is $\mathrm{End}(\emptyset_\Sigma)$, the endomorphism ring of $\emptyset_\Sigma \in tl(\Sigma)$.

So it is spanned by ~~t~~ ribbon links in $\Sigma \times I$, modulo the skein relations $\times = q \asymp + q^{-1})($, $\bigcirc = -q^2 - q^{-2}$. You can project any of these down to Σ, resolve crossings and get a non-crossing diagram in Σ:

 $\in \mathrm{End}(\emptyset_\Sigma)$.

These are a basis for $\mathrm{End}(\emptyset_\Sigma)$.

A category with a distinguished object is called an E_0 category or pointed category. There are various kinds of functors between E_0 categories. Here's the best kind:

Defn: The 2-category $E_0(CAT/\mathbb{Z}_q)$ has

- Objects = $\mathbb{Z}_q$-linear categories $\mathcal{C}$ each equipped with a distinguished object $C \in \mathcal{C}$.
- 1-morphisms = $\hom((\mathcal{C}, C), (\mathcal{D}, D))$

  $\quad\quad\quad\quad\quad\quad \shortparallel$

  $\mathbb{Z}_q$-linear functors $F : \mathcal{C} \to \mathcal{D}$ along with a map $f : D \to F(C)$ in $\mathcal{D}$.
- 2-morphisms = $\hom((F,f),(G,g))$

  $\quad\quad\quad\quad\quad\quad \shortparallel$

  natural transformations $\eta : F \to G$ s.t.

$$\begin{tikzcd} & F(C) \arrow[d, "\eta(C)"] \\ D \arrow[ur, "f"] \arrow[r, "g"'] & G(C) \end{tikzcd}$$

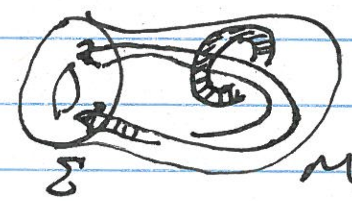
commutes.

Getting the direction of f right is the hard part.

E.g. Let $*$ = the one-element category, with its unique distinguished object $*$. (Linearize: $\hom(*,*) = \mathbb{Z}_q$.)

Then $\hom((*,*), (\mathcal{C}, C)) = \{(X, x)$ where $X \in \mathcal{C}$ and $x : C \to X\}$.

What about 3-manifolds? Let M have $\partial M = \Sigma$. For each configuration $X \in t\ell(\Sigma)$, we can consider the $\mathbb{Z}_q$-module of ribbon tangles in M ending at X, modulo the skein relations.



Now it has no reason to be a free $\mathbb{Z}_q$-module.
~~[illegible]~~.
Up to fixing some orientation conventions, the following should be obvious:

<u>Thm</u>: The assignment $X \mapsto$ this module defines a ~~[illegible]~~ $\mathbb{Z}_q$-linear functor

$$t\ell(\Sigma)^{op} \longrightarrow \mathbb{Z}_q\text{-MOD}.$$

<u>Defn</u>: The space of $\mathbb{Z}_q$-linear functor $\mathcal{C}^{op} \to \mathbb{Z}_q$-MOD, where ~~$\mathbb{Z}_q$~~ $\mathcal{C}$ is a small $\mathbb{Z}_q$-linear category, is called $\mathrm{PSH}(\mathcal{C})$ or $\mathrm{MOD}(\mathcal{C}^{op})$ — the "$\mathcal{C}^{op}$-modules" or "presheaves on $\mathcal{C}$".

<u>Thm (Yoneda)</u>: $\mathrm{PSH}(\mathcal{C})$ is the free $\mathbb{Z}_q$-linear cocompletion of $\mathcal{C}$. i.e.:

- it emits an embedding $\mathcal{C} \hookrightarrow \mathrm{PSH}(\mathcal{C})$
  $X \longmapsto \hom(\cancel{*}\, -, X)$.
- PSH has all colimits.
- Every object of PSH is a colimit of objects of the form $\hom(-, X)$
- If $\mathcal{D}$ has all colimits and is $\mathbb{Z}_q$-linear,

then every $\mathbb{Z}_t$-linear functor $\mathcal{C} \to \mathcal{D}$ extends to a $\mathbb{Z}_t$-linear <u>colimit</u>-pre<u>serving</u> functor $\mathrm{PSH}(\mathcal{C}) \to \mathcal{D}$.

Let's define $\mathrm{TL}(\Sigma) = \mathrm{PSH}(\mathit{tl}(\Sigma))$. So the Yoneda embedding gives $\mathit{tl}(\Sigma) \hookrightarrow \mathrm{TL}(\Sigma)$, and in particular $\emptyset_\Sigma \in \mathrm{TL}(\Sigma)$, and for any $X \in \mathit{tl}(\Sigma)$ and $F \in \mathrm{TL}(\Sigma)$, we have

$$\hom(X, F) = F(X) \in \mathbb{Z}_t\text{-MOD}.$$

Let's define $\mathrm{TL}(M)$ to be the functor from above, so $\mathrm{TL}(M) \in \mathrm{TL}(\Sigma)$.

What is $\hom(\emptyset_\Sigma, \mathrm{TL}(M)) = \mathrm{TL}(M)(\emptyset_\Sigma)$? It is the $\mathbb{Z}_t$-module spanned by links in M. This is called the <u>skein module</u> of M. It is acted on by the skein algebra $\mathrm{End}(\emptyset_\Sigma)$. (By the Yoneda theorem, $\mathrm{End}_{\mathit{tl}}(\emptyset) = \mathrm{End}_{\mathrm{TL}}(\emptyset)$.)

But this skein module has a distinguished element $\emptyset_M \in \hom(\emptyset_\Sigma, \mathrm{TL}(M))$, namely the empty link. So

$$(\mathrm{TL}(M), \emptyset_M) \in \mathrm{Hom}_{E_0(\mathrm{CAT})}(*, (\mathrm{TL}(\Sigma), \emptyset_\Sigma)).$$

You can also play this game in lower dimensions:

$\mathrm{TL}(S^1) = \mathrm{PSH}(\mathit{tl}(S^1))$ is monoidal.

$\mathrm{TL}(pt) = \mathrm{PSH}(\mathit{tl}) =$ is braided monoidal.

Thm (joint w/ C. Scheimbauer)

There is an $(\infty,4)$-category $E_2(LPCAT/\mathbb{Z}_q)$ with

- 0-morphisms = $\mathbb{Z}_q$-linear ~~LP~~ braided monoidal categories.
- 1-morphisms = $\mathbb{Z}_q$-linear LP monoidal "bimodule" categories.
- 2-morphisms = $\mathbb{Z}_q$-linear LP pointed (E_0) "bimodule" categories.
- 3-morphisms = $\mathbb{Z}_q$-linear LP E_0 "intertwiner" functors.
- 4-morphisms = E_0 natural transformations.

By "E_0" I mean as in $E_0(CAT/\mathbb{Z}_q)$. "bimodule" and "intertwiner" are categorifications of what you are familiar with from ring theory. "LP" means "Locally presentable": it says that all categories involved should be cocomplete (+ conditions on a generating set) and all functors should be cocontinuous.

Thm (me): TL is an oriented fully-extended 3-D TQFT valued in $E_2(LPCAT/\mathbb{Z}_q)$.

This means: each oriented compact k-manifold N with corners, for $k \leq 3$, is assigned a k-morphism $TL(N)$ (or $(TL(N), \emptyset_N)$) in $E_2(LPCAT/\mathbb{Z}_q)$

Compositions are well-behaved: there are coherent equivalences

$TL($  $) \circ TL($  $) \cong TL($  $)$.

Thm (me): TL is robust under changing coefficients $\mathbb{Z}_q \to R$ for other com. rings R.
If you use $R = \mathbb{C}$ with $q \mapsto 1$, or -1, then

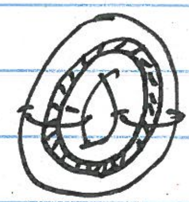
$$TL \underset{\mathbb{Z}_q}{\otimes} \mathbb{C}_{q=1} : \quad N \mapsto QCoh(Loc_{SL(2)}(N))$$

if $\dim N \leq 2$.

$$N \mapsto \mathcal{O}(Loc_{SL(2)}(N))$$

if $\dim N = 3$.

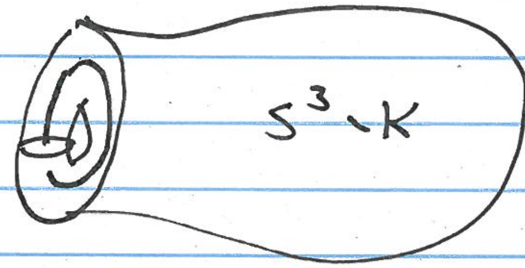
To end, let me come back to the Jones polynomial.
~~Consider TL(~~
There is an element $|1\rangle \in \hom_{TL(T^2)}(\emptyset_{T^2}, T(D^2 \times S^1))$ corresponding to a ribbon going once along the core of the solid torus $D^2 \times S^1$:



Let $K : S^1 \hookrightarrow S^3$ be a knot, and $M = S^3 \setminus K$ its complement, which has torus boundary.
You can look at the image of $|1\rangle \otimes \emptyset_M$ in

$$TL(D^2 \times S^1) \underset{TL(T^2)}{\circ} TL(S^3 \setminus K) = TL(S^3) = \mathbb{Z}_q.$$

This is precisely the Jones polynomial of K.

TL(

) = $J(K) \in \mathbb{Z}_q$.

If you know what the colored Jones polynomial is, you can get it too. So TL knows <u>everything</u> about (the SL(2) part of) quantum topology.