Asymptotics of oscillating integrals via homological perturbation theory

Theo Johnson-Freyd, GRASP seminar, UC Berkeley, 14 Oct '11.

X = compact (for now) oriented (for convenience) manifold.

equipped with $\mu \in \Omega^{top}_{dR}(X)$ nowhere-vanishing (for now).

We are interested in expectation values

$$\mathcal{C}^\infty(X) \to \mathbb{R} \qquad f \mapsto \langle f \rangle_\mu = \frac{\int_X f\mu}{\int \mu}.$$

Observation: Consider chain complex $(\Omega^{top-\bullet}(X), d_{dR})$. (Homological grading: $|d| = -1$).

$\int : \Omega^{top-\bullet} \to \mathbb{R}$ is a chain map.

if X = connected, $H^0(\Omega^{top-\bullet}, d) = 1$-dim, so $\int$ is determined up to constant by being a chain map. In general, $\int$ is determined by finitely much data.

$$C^\infty(X) \xrightarrow{\times\mu} \Omega^{top-\bullet} \xrightarrow{\int} \mathbb{R}$$

Question: Is $C^\infty(X)$ the deg-0 part of a complex iso to $\Omega^{top-\bullet}$? If so, $\int\mu : C^\infty(X) \to \mathbb{R}$ is determined up to finitely much data by being a chain map.

Answer: Yes.

$$MV^\bullet = \Gamma(T^{\wedge\bullet}X)$$

$$\begin{tikzcd}
 & 0 & 1 & 2 \\
= & C^\infty(X) \arrow[d, "\mu"] & \text{Vector fields} \arrow[d, "\mu"] & \cdots \\
 & \Omega^{top} & \Omega^{top-1} \arrow[l, "d"'] & \cdots \arrow[l]
\end{tikzcd}$$

Can define "contract with μ": $MV^\bullet \to \Omega^{top-\bullet}$.

Set $\Delta_\mu = \mu^{-1} \circ d \circ \mu$. Then

$$\int\mu : (MV^\bullet, \Delta_\mu) \to \mathbb{R}$$

is a chain map.

$\langle\cdot\rangle_\mu$ is too. It satisfies $\langle 1\rangle_\mu = 1$.

If X = connected, $\langle\cdot\rangle_\mu$ is determined by Δ_μ.

$MV^\bullet$ = (super) com alg under $\wedge$

$= C^\infty(\pi T^* X)$.

but $(MV^\bullet, \Delta_\mu)$ is not cdga, because

Δ_μ is not a derivation.

Fact: • Δ_μ is second-order differential operator.

• $\pi T^* X$ is "symplectic", i.e.

$MV^\bullet$ has (non-deg) "Poisson"

bracket $\mathcal{P}$ = Schouten-Nijenhuis bracket, $|\mathcal{P}| = -1$.

• principal symbol of $\Delta_\mu = \mathcal{P}$.

• Δ_μ is derivation of $\mathcal{P}$.

Proof: Easy calculation in local coords.

Defn: A BV-laplacian is $\Delta_m : MV^\bullet \to MV^{\bullet-1}$

s.t. $\frac{1}{2}[\Delta_m, \Delta_m] = \Delta_m^2 = 0$.

$$[\Delta_m, m] = \mathcal{P}$$

$$[\Delta_m, \mathcal{P}] = 0.$$

Cor: {BV Laplacians} is affine, modeled on {symplectic vector fields}.

Cor: Under $\mu \mapsto e^s \mu$,

$$\Delta_\mu \mapsto \Delta_\mu + \text{some symp v-field.}$$

fact: is Hamiltonian $= \mathcal{P}(s, -)$.

Thm (Koszul, ~85)

$$\begin{tikzcd} \{\text{BV Laplacians}\} \arrow[rr, equal, "\text{canonical}"] & & \{\text{flat connections on } (T^*)^{\wedge top}\} \\ & \{\text{measures} = \text{trivializations of } (T^*)^{\wedge top}\} \arrow[ul, "\Delta_\mu \mapsfrom \mu"] \arrow[ur] \arrow[u, phantom, "\circ"] & \end{tikzcd}$$

flat $= (\Delta_\mu^2 = 0)$.

$$\partial = \Delta_{e^{s/\hbar}\mu} = \frac{1}{\hbar} P(s,-) + \Delta_\mu .$$

$H^\bullet$ doesn't change under rescaling $\partial \to \hbar\partial$ if $\hbar$ is invertible.

When $\hbar \approx 0$, it feels better to write

$$P(s,-) + \hbar \Delta_\mu . \leftarrow \text{a differential on } MV^\bullet(X).$$

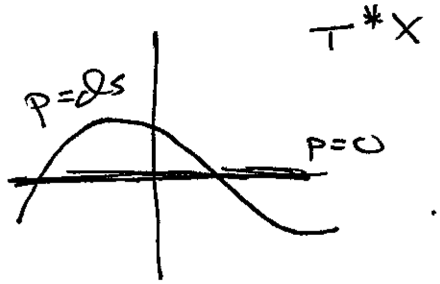
<u>Remark</u>: $P(s,-) =$ "contract with ds".

Let's study $\left(MV^\bullet(X),\ P(s,-)\right)$. This is cdga, i.e. $= \mathcal{O}(\text{derived affine scheme})$.

$$= \mathcal{O}(\pi T^*X, \text{ some v-field}).$$

$$H^0(\pi T^*X, P(s,-)) = \mathcal{O}(ds = 0).$$

<u>Fact</u>: $\left(\pi T^*X,\ P(s,-)\right) =$ "derived intersection"



$$= \mathcal{O}(\{p=0\}) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*X)} \mathcal{O}(\{ds=p\})$$

$$\mathcal{O}(X)$$
$$\parallel$$
Reason: $\mathcal{O}(\{p=0\})$ has Koszul resolution over $\mathcal{O}(T^*X)$ given by

$$\mathcal{O}\big((T^* \oplus \pi T^*)X\big), \quad d: T^* \xrightarrow{\sim} \pi T^*.$$

[Costello]

Fact: In any symp man, derived intersection of Lags is "symp" $|P| = -1$.

So $\hbar \to 0$ asymptotics of $\langle f \rangle_{e^{s/\hbar}\mu} = \dfrac{\int f e^{s/\hbar}\mu}{\int e^{s/\hbar}\mu}$

should be controlled by

$$(MV^\bullet, P(s,-) + \hbar \Delta_\mu)$$

where $\hbar$ = formal variable.

e.g.: $X = \mathbb{R}^n$, s has unique non-deg crit. point.

$$H^\bullet(MV^\bullet, P(s,-)) = \underset{0}{\mathbb{R}} \quad \underset{1}{0} \quad \cdots \quad \underset{2}{0}$$

Spectral sequence $\Rightarrow H^\bullet(MV^\bullet, P(s,-) + \hbar\Delta)$
$= \mathbb{R}$.

So $\langle f \rangle \in \mathbb{R}[\hbar]$ is the unique soln

to $\quad f \equiv \langle f \rangle$ mod exact in

$$\left(\mu V^{\circ}(\mathbb{R}^n),\ P(s,-) + \hbar \Delta \right).$$

When $\mu =$ Lebesgue, can solve in coords.

$$P(s,\ \varphi^i(x)\xi_i) + \hbar \Delta(\varphi^i \xi_i))$$

$\varphi^i(x)\frac{\partial}{\partial x^i} \in \Gamma(T\mathbb{R}^n)$

$\updownarrow$

$\varphi^i(x)\xi_i \in \mu V^1.$

$$= s^{(2)}_{ij}\ \varphi^i(x)\, x^j$$

$$+ \text{``}\sum_{n \geq 3} s^{(n)}\ \varphi(x) \cdot x^{n-1}\text{''}$$

$$+ \hbar \cdot \frac{\partial \varphi^i(x)}{\partial x^i}.$$

So $\cancel{s^{(2)}}$

$$f(x) = s^{(2)}_{ij}\ \varphi^i(x)\ x^j \equiv \text{higher deg in } x$$

$$+ \text{higher deg in } \hbar$$

$\uparrow$ can be ~~anything~~ any $f(x) =$ Linear + higher.

You can solve combinatorics directly.

Fact (–, Owen Gwilliam):

when you do, you get standard Feynman diagram expansion.

Here's a more robust machine.

Choose homotopy equiv

$$L^\bullet = H^\bullet(\mathcal{M}V^\bullet, \mathcal{O}(s,-)) \underset{i}{\overset{p}{\rightleftarrows}} \underbrace{(\mathcal{M}V^\bullet, \mathcal{O}(s,-))}_{\overset{\shortparallel}{(\mathcal{M}^\bullet, \partial)}}, \ \varphi$$

i.e. i, p are chain maps $p \circ i = \mathrm{id}$,

$i \circ p = \mathrm{id} - [\partial, \varphi]$.

+ "side conditions" $\varphi^2 = 0$, $\varphi i = 0$, $p\varphi = 0$.

<u>Thm</u>: if $\cdot (\partial + \delta)^2 = 0 \quad (\delta : \mathcal{M}^\bullet \to \mathcal{M}^{\bullet-1})$

Cold; good reference is Crainic '04].

~~[illegible]~~

$\cdot (1 - \delta\varphi)$ is invertible, then the following is a homotopy equiv:

$$\begin{tikzcd} \left(L^\bullet, p\circ\delta(1-\varphi\delta)^{-1}\circ i\right) \arrow[dr, bend right, "(1-\varphi\delta)^{-1}\circ i"'] & \\ & \left(M^\bullet, \partial+\delta\right) \arrow[ul, bend right, "p\circ(1-\delta\varphi)^{-1}"'] \arrow[loop right, "\varphi(1-\delta\varphi)^{-1}"] \end{tikzcd}$$

<u>Proof</u>: direct calculation.

<u>Application</u>: ~~[scribbled]~~ Allows to compute

~~[scribbled]~~ $H^\bullet(\mathcal{M}V^\bullet, \mathcal{P}(s,-)+\hbar\Delta)$

to all orders in $\hbar$.

Best choice

if $\{ds = p\} \cap \{0 = p\}$ is transverse,

then $\{ds = 0\}$ is closed subman, and

$$C^\infty(\pi T^* \{ds=0\}) \xleftarrow{\text{restrict}} \big(C^\infty(\pi T^* X),\ P(s,-)\big)$$

is chain map. ~~this~~ $\overset{\shortparallel}{H^\bullet(P(s,-))}$

Choose splitting. $L^\bullet \underset{}{\overset{\text{restrict}}{\leftrightarrows}} M^\bullet \circlearrowleft$

↳ corresponds to $\{ds=0\} \leftrightarrows X$ trivializing tubular nbhd.

HPT tells you:
- deformation of "restrict"
  ↳ corresponds to "integrate out fibers"
- "measure" on $\{ds = 0\}$
  ↳ new differential on $L^\bullet$.

The point is that the homological algebra (often) makes sense in ∞ dimensions, stacks, ...
except for \$64,000 question: Find a BV Laplacian.

any BV Laplacian $\longrightarrow$ "asymptotic expectation values for oscillating integrals".