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Homological Perturbation and Factorization Algebras

26 ~~Ma~~ May 2011, Northwestern University

Based on conversations with Josh Shadlen and Owen Gwilliam.

Definition

A special deformation retract ~~m~~ (SDR) is

$$\begin{tikzcd} (L, d_L) \arrow[r, bend right, "i"'] & (M, d_M) \arrow[l, bend right, "p"'] \arrow[loop right, "\phi"] \end{tikzcd} \qquad (*)$$

where:

- (L, d_L) and (M, d_M) are chain complexes, i.e. $\mathbb{Z}$-graded abelian groups (modules, ...) with $|d_L| = |d_M| = +1$, $d_L^2, d_M^2 = 0$
- i, p are chain maps, i.e. $[d., i] = 0$ $[d., p] = 0$.
- ~~so~~ $p \circ i = \mathrm{id}_L$, $\quad i \circ p = \mathrm{id}_M - [d, \phi]$

  ↑ signed commutator $= d\phi + \phi d$, since $|\phi| = -1$.
- $\phi^2 = 0$, $p \circ \phi = 0$, $\phi \circ i = 0$.

Example:

Work over ring $\supseteq \mathbb{Q}$. Given SDR as in $(*)$, build

$$\begin{tikzcd} \left(\mathrm{Sym}\, L,\ d_L \text{ extended as a derivation}\right) \arrow[r, bend right, "I"'] & \left(\mathrm{Sym}\, M,\ d_M \text{ extended as a derivation}\right) \arrow[l, bend right, "P"'] \arrow[loop right, "\Phi"] \end{tikzcd}$$

↖ signed sym

where:

- I, P = extend i, p as homomorphisms (or $\widehat{\mathrm{Sym}}$ or $\widetilde{\mathrm{Sym}}$ or ...)

How to understand Φ? Use i, p to split M as direct sum

$$M = L \oplus N \qquad \mathrm{Proj}_L = i \circ p, \quad \mathrm{Proj}_N = (1 - i \circ p).$$

"ghosts" (on N)

Then $\mathrm{Sym}\, M = \mathrm{Sym}\, L \otimes \mathrm{Sym}\, N$. Extend Proj_N to a derivation $\underline{n}$ = "ghost number". Extend ϕ to a derivation. Set:

$$\Phi = \begin{cases} 0 & \text{on } \ker \underline{n} \\ \frac{1}{n}\left(\text{extension of } \phi \text{ as a der}\right) & \text{on } \ker(\underline{n} - n), \quad n \in \mathbb{Z}_{>0}. \end{cases}$$

$$= \text{"} \int_{t=0}^{1} \phi(-) \Big|_{\substack{\text{rescale } N \\ \text{by } \times t}} \frac{dt}{t} \text{"}.$$

<u>Theorem</u> (from 1960s, best proof in Crainic '04).

Given SDR $(*)$, suppose $\delta = \delta_M$ is a "small perturbation" of d_M. I.e. $|\delta| = +1$, and:

$$(d_M + \delta)^2 = 0 \qquad \text{"quantum master equation"}$$

$$(i d_M - \delta \phi)^{-1} \text{ exists} \qquad (\text{equiv. } [i d_M - \phi\delta]^{-1} \text{ exists}).$$

Then the following is a SDR:

$$(L, d_L + \delta_L) \underset{i_\delta}{\overset{p_\delta}{\rightleftarrows}} (M, d_M + \delta) \circlearrowleft \phi_\delta$$

where:

$$\delta_L = p \circ \left(\delta\,(i\partial - \phi\delta)^{-1}\right) \circ i = p \circ \left((i\partial - \delta\phi)^{-1}\delta\right) \circ i$$

$$\phi_\delta = \phi\,(i\partial - \delta\phi)^{-1} = (i\partial - \phi\delta)^{-1}\phi$$

$$p_\delta = p \circ (i\partial - \delta\phi)^{-1}$$

$$i_\delta = (i\partial - \phi\delta)^{-1} \circ i .$$

(SDR conditions $\Rightarrow$ can replace $\delta\phi$, $\phi\delta$ by $[\delta,\phi]$ throughout)

<u>Example</u>:

X has basis $x^1, \ldots, x^n$. $[1]X^*$ has dual basis $\xi_1, \ldots, \xi_n$.
↖shift down one degree

Pick $a^{ij} : \mathbb{K} \to X \otimes X$ symmetric, $X^* \xrightarrow{\sim} X$ is invertible.

Use to build

$$(M, \partial_M) = [1]X^* \xrightarrow[a]{\sim} X .$$

$(L, \partial_L) = (0,0)$. $a^{-1} \rightsquigarrow \phi$. i, p are zero map. Build

$$\mathbb{K} = (\widehat{\mathrm{Sym}}(0), 0) \leftrightarrows (\widehat{\mathrm{Sym}}\, M\partial, \partial_M) \circlearrowleft$$

power series in $x^1, \ldots, x^n, \xi_1, \ldots, \xi_n$.

$$\partial = \sum_{i,j} a_{ij}\, x^i \frac{\partial}{\partial \xi_j} .$$

Now pick power series $b(x) \in \widehat{\mathrm{Sym}}(X) = \mathbb{K}[\![x^1, \ldots, x^n]\!]$ vanishing like a cubic.

~~[scribble]~~ Perturbation $\delta = \sum_i \frac{\partial b}{\partial x^i} \frac{\partial}{\partial \xi_i} \Rightarrow$ only $\phi \rightsquigarrow \phi_\delta$ deforms.

Now extend $[\![\hbar]\!]$, $|\hbar| = 0$. Perturb again:

$$\delta' = \sum_i \hbar \frac{\partial^2}{\partial x^i \partial \xi_i} .$$

Now p also deforms.

<u>Fact</u>: If $f \in \widehat{\mathrm{Sym}}(X) = \mathbb{K}[\![x^1, \dots, x^n]\!]$, and $\mathbb{K} = \mathbb{R}$, and $\hbar \to 0$ along "pure imaginary" axis, then

$$p(f) \overset{\in \mathbb{K}}{=} (\hbar \to 0 \text{ asymptotics of oscillating integral}) \quad \frac{\displaystyle\int_{x^1=-\infty}^{\infty} \cdots \int_{x^n=-\infty}^{\infty} f(x)\, e^{\frac{1}{\hbar}\left(\frac{ax^2}{2} + b(x)\right)} dx}{\displaystyle\int_{-\infty}^{\infty} \cdots \int e^{\frac{1}{\hbar}\left(\frac{ax^2}{2} + b(x)\right)} \underset{\uparrow \text{Lebesgue}}{dx}}.$$

<u>Proof</u>: This complex is a "twisted de Rham complex" for space X^*, twisted by measure $e^{\frac{1}{\hbar}(-)} dx$. If you use $\mathcal{C}^\infty_{cpt}$ rather than $\widehat{\mathrm{Sym}}$, you can do similar things with $\hbar$ non-formal.

<u>Example:</u> Move to one-dimensional example. V = vector space. $\mathcal{U} \subseteq \mathbb{R}$ is open interval.

$$\begin{tikzcd} (V, d_V = 0) \arrow[r, bend right, "\delta"'] & \left([1]\Omega^\bullet_{cpt}(\mathcal{U}) \otimes V, \underset{\uparrow \text{de Rham}}{\partial}\right) \arrow[l, bend right, "p"'] \end{tikzcd}$$

$p =$ "integrate top forms": $[1]\,\Omega^1_{cpt}(U) \otimes V \to \mathbb{R} \otimes V = V$.

i depends on choice: choose $u \in [1]\,\Omega^1_{cpt}(U)$ with $\int u = 1$, and $i(v) = u \otimes v$ for $v \in V$.

<u>the homotopy is ϕ</u>:

- is zero on $[1]\,\Omega^0_{cpt}$.
- if $\alpha \in [1]\,\Omega^1_{cpt} \otimes V$, then

$$\int \left(\alpha - u \cdot \int \alpha\right) = 0, \quad \text{since } \int u = 0.$$

Since U is interval, $\Rightarrow \partial^{-1}\left(\alpha - u \int \alpha\right)$ exists.

$\phi = \partial^{-1}\left(\mathrm{id} - u \int\right)$.

Now move to

$$\mathrm{Sym}(V) \rightleftarrows \mathrm{Sym}\left([1]\,\Omega^{\cdot}_{cpt}(U) \otimes V\right) \circlearrowleft \Phi$$

Let's work out homotopy Φ on Sym^2.

$u \otimes \int$, $\quad \mathrm{id} - u \otimes \int$

Steps: projections ip, $\mathrm{id} - ip$ decomposes Sym into "particles" and "ghosts" and Φ divides by ghost number.

$$\alpha \odot \beta = \left(u\textstyle\int\alpha + (\alpha - u\int\alpha)\right) \odot \left(u\int\beta + (\beta - u\int\beta)\right)$$

$$= \quad u\textstyle\int\alpha \odot u\int\beta \qquad \underline{n} = 0.$$

$$+ \; u\textstyle\int\alpha \odot (\beta - u\int\beta) + (\alpha - u\int\alpha) \odot u\int\beta \qquad \underline{n} = 1$$

$$+ \; (\alpha - u\textstyle\int\alpha) \odot (\beta - u\int\beta) \qquad \underline{n} = 2$$

$\Phi =$ "act by $\partial^{-1} \circ (i\partial - u\int)$ as a derivation, and divide by $\underline{n}$".

$$\Phi(\alpha \cdot \beta) = \quad 0 \qquad \underline{n} = 0$$

$$+$$

$$u\textstyle\int\alpha \cdot \partial^{-1}(\beta - u\int\beta) + \partial^{-1}(\alpha - u\int\alpha) \cdot u\int\beta \qquad \underline{n} = 1$$

$$+ \qquad\qquad\qquad +$$

$$0 \cdot \;\cdots \quad + \quad \cdots \cdot 0$$

$$+$$

$$\frac{1}{2}\left(\partial^{-1}(\alpha - u\textstyle\int\alpha) \cdot (\beta - u\int\beta) + (\alpha - u\int\alpha) \cdot \partial^{-1}(\beta - u\int\beta)\right).$$

$$= \frac{1}{2}\left(\partial^{-1}(\alpha - u\textstyle\int\alpha) \cdot (\beta + u\int\beta) + (\alpha + u\int\alpha) \cdot \partial^{-1}(\beta - u\int\beta)\right). \qquad (**)$$

OK, let's try a perturbation. Pick

$\langle , \rangle : V \otimes V \to \mathbb{R}$ antisymmetric,

and try $\delta = \hbar \int \langle \wedge \rangle$, extended as 2nd-order operator

Then $p_\delta = p \circ (\mathrm{id} + \delta\Phi + \dots)$.

On $\mathrm{Sym}^0, \mathrm{Sym}^1$, δ vanishes, so $p_\delta = p = \int$. On Sym^2,

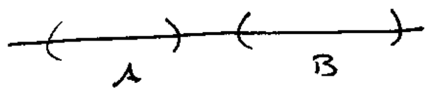
$$p_\delta = p \circ (\mathrm{id} + \hbar \int \langle \wedge \rangle \circ \Phi):$$

← depends on u!

$$\alpha \cdot \beta \mapsto \int \alpha \cdot \int \beta \quad + \hbar \int \langle (**) \rangle .$$

$\uparrow$ $\mathrm{Sym}^2(V)$

<u>Note</u>: $i_\delta = i$, because $\delta \circ i = 0$. Similarly $\delta d|_{\mathrm{Sym}(V)}$ does not change.

Use this to define new (homotopy) associative product $\star$ on $\mathrm{Sym}(V)$: pick 

$\mathbb{R} = U$ $\quad u \in \Omega^1_{cpt}(\mathbb{R})$

$a \in \Omega^1_{cpt}(\mathbb{R}_{<0})$

$b \in \Omega^1_{cpt}(\mathbb{R}_{>0})$

$\int u = \int a = \int b = 1$.

Define $f \star g$, for $f, g \in \mathrm{Sym}(V)$, by:

(1) • include $f \longmapsto \mathrm{Sym}([1]\Omega^\bullet_{cpt}(\mathbb{R}) \otimes V)$ along a.
• include g along b.

(2) Sym-multiply in $\mathrm{Sym}([1]\Omega^\bullet_{cpt}(\mathbb{R}) \otimes V)$.

(3) project along u.

Let's calculate when $f, g \in \mathrm{Sym}^1$. Then

$$f \star g = \underbrace{\int(a \otimes f) \cdot \int(b \otimes g)}_{f \cdot g} + \hbar \int \dots$$

$$= f\cdot g + \frac{\hbar}{2}\int_{\mathbb{R}} \left(\partial^{-1}(a-u{\textstyle\int} a)\wedge (b+u{\textstyle\int} b) + (a+u{\textstyle\int} a)\wedge \partial^{-1}(b-u{\textstyle\int} b)\right) \langle f,g\rangle$$

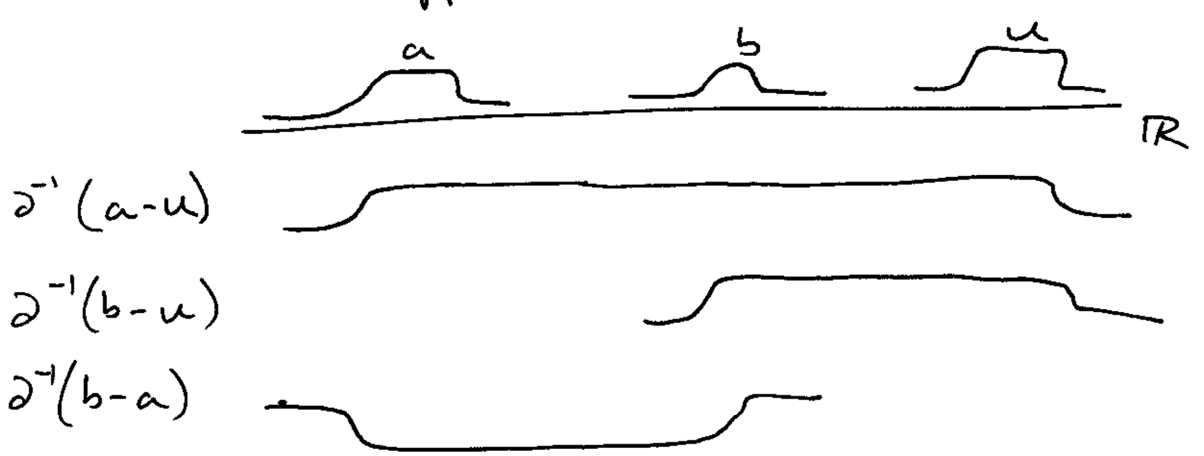
$\int a = 1 = \int b$. How to calculate

$$\int_{\mathbb{R}} \left(\partial^{-1}(a-u)\wedge(b+u) + (a+u)\wedge\partial^{-1}(b-u)\right)$$

$$= \int_{\mathbb{R}} \left(\partial^{-1}(a-u)\wedge b + a\wedge\partial^{-1}(b-u) + u\wedge\partial^{-1}(b-a) \right) \quad ?$$

Change $u \rightsquigarrow u+\partial(u')$. Check that $\int$ does not
$a \rightsquigarrow a+\partial(a')$ change.
$b \rightsquigarrow b+\partial(b')$

Then can suppose



Only $\partial^{-1}(a-u)\wedge b = b$ is non-zero. $\int b = 1$.

So $f \star g = f\cdot g + \frac{\hbar}{2}\langle f,g\rangle$.

$\Rightarrow (\mathrm{Sym}(V), \star)$ is the Weyl algebra.

<u>Example:</u> Same set-up. Pick Lie bracket

$$[,]: V \otimes V \to V.$$

Set $\delta = \hbar [\wedge]$ extended as 2^{nd}-order diff op.

$[\partial, \delta] = 0$. $\delta^2 = 0$ iff Jacobi.

Same argument to construct $\star$ product. On linears,

$$f \star g = f \cdot g + \frac{\hbar}{2}[f,g].$$

$$(\mathrm{Sym}(V), \star) = \text{Universal enveloping alg.}$$

<u>Discussion:</u> $\mathrm{Sym}(V)$ = functions on V^*, and $\langle , \rangle$ or $[,]$ induces a Poisson structure on V^* (either translation invariant or linear in coordinates).

The field theory is some model for "topological quantum mechanics with phase space $= V^*$" even though V^* is not symplectic.

Two generalizations are worth highlighting.

(1) If poisson structure on V^* has quadratic + higher part, then $\Omega^\bullet_{cpt}$ model isn't quite good enough: it does not have comultiplication dual to $\wedge : \Omega^\bullet \otimes \Omega^\bullet \to \Omega^\bullet$. Fixing this requires "renormalization theory".

(2) V^* itself might be a dg manifold. Then you might have an extra dg structure on chains

$$\mathrm{Sym}(V), Q_V \xleftarrow{\mathrm{Sym}(\int)} \mathrm{Sym}(\Omega^\bullet_{cpt} \otimes V), Q_V + \partial$$

so that this is dg map. But "insert at u" will not be dg map usually: fix by turning on Q_V with a coupling constant, and again run homological perturbation lemma.

<u>Example</u>

A particular case is when $V^* = T^*X$ for a manifold X. Then as above $\mathrm{Sym}(V) \rightsquigarrow$ Weyl alg. But if X has symmetric, degree 1 bivector field ("P_0 manifold") $\rightsquigarrow$ dg structure on T^*X.

$\rightsquigarrow$ factorization alg on $\mathbb{R}_{\geq 0}$ quantizing "fields valued in T^*X that send $0 \mapsto X \subseteq T^*X$".

In practice: $\mathrm{Sym}(\,\Omega^\bullet(\mathbb{R}_{\geq 0}) \otimes (X \oplus X^*))$

~~cpt~~ cpt-support except the $\Omega^0 \otimes X$ part only has to be <u>constant</u> near 0, not $\emptyset$.

Then ~~[illegible]~~ quantization $\rightsquigarrow$ E_1-algebra for $\mathbb{R}_{>0}$
E_0-module thereof.

Study the E_0 module. In pieces, first say

$$\mathcal{C}[1]\Omega^{\bullet}_{\ldots}(\mathbb{R}_{>0}) \otimes (X \oplus X^*) \rightleftarrows \overset{\mathrm{ev}_0}{[1]X} \oplus \overset{\int}{X} \oplus \overset{\int}{X^*},$$

the extra bit of chain

with some residual differential $[1]X \xrightarrow{\sim} X$ left over from de Rham d.

Then quantizing

$$\delta = \hbar\Delta \rightsquigarrow \hbar \cdot \text{pairing } [1]X \otimes X^* \to \mathbb{K}.$$

Now
$$\mathrm{Sym}(X^*) \hookrightarrow \mathrm{Sym}([1]X \oplus X \oplus X^*) \circlearrowleft$$