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Topological Duality of Hopf Algebras

Workshop on Cluster Algebras and Lusztig's Semicanonical Bases

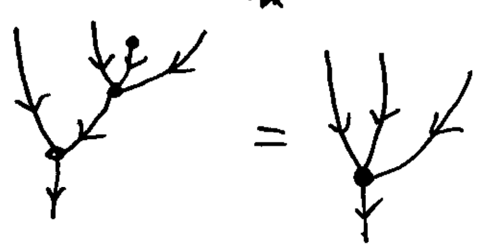
Eugene, OR, 13 June 2011

Since this is the first talk, it might be our only chance to have something completely elementary and understandable. So please interupt with questions.

§ Groups.

Let $\mathcal{C}$ be a category with finite products (including empty product = terminal object = "pt").

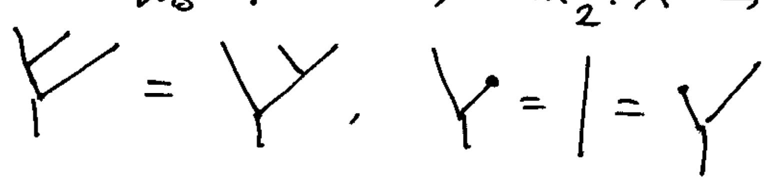
<u>Defn</u>: A <u>monoid in $\mathcal{C}$</u> is an object $X \in \mathcal{C}$ along with, for each $n \in \mathbb{N}$, a map $m_n: X^{\times n} \to X$; such that for every planar rooted tree, the corresponding composition of "m"s is $m: X^{\# \text{ of leaves}} \to X$.



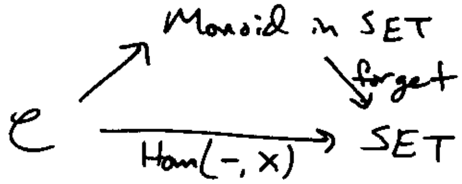
<u>Lemma</u>: TFAE:

(i) X is equipped with a monoid structure $m_0, m_1, m_2, \dots$

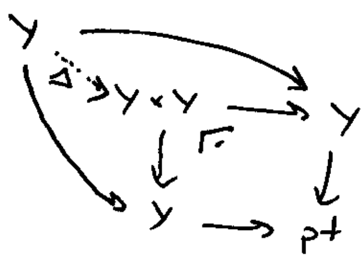
(ii) X is equipped with $m_0: pt \to X$, $m_2: X^{\times 2} \to X$ such that



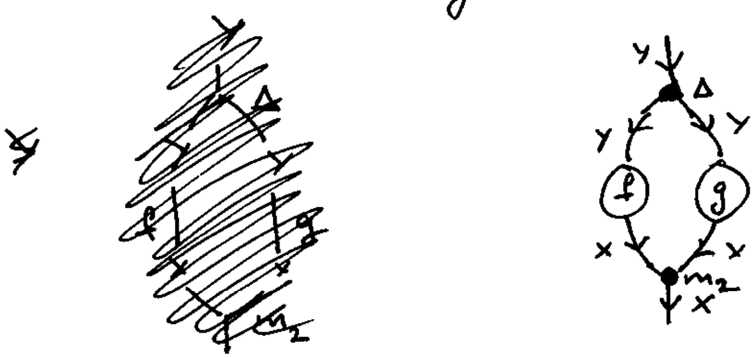
(iii) The representable sheaf $\mathrm{Hom}_{\mathcal{C}}(-, X)$ is equipped with a factorization



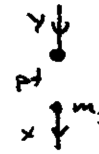
Remark: Property (iii) uses that we are working with cartesian products and not some more general monoidal structure. If $Y \in \mathcal{C}$, it has a canonical diagonal map $\Delta : Y \to Y^{\times 2}$ via



Given $f, g \in \mathrm{Hom}(Y, X)$, their convolution product is $f \star g : Y \to X$ given by

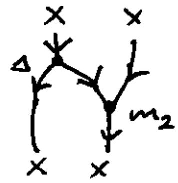


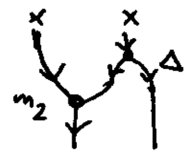
Exercise: This is associative. The unit is



, where $Y \to pt$ is the unique such map.

Lemma: Let $X, \dots$ be a monoid. TFAE:

(i) The map  is invertible.

(i′) The map  is invertible.

(ii) The identity map $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ is $\star$-invertible.

(iii) $\mathrm{Hom}(-,X)$ factors through {Groups in SET}.

Defn: If X has properties (i–ii) above, it is a *group*.

Examples:
- If $\mathcal{C} = \mathrm{SET}$, then ~~groups = groups~~. groups = discrete groups.
- If $\mathcal{C} = \mathrm{Manifolds}$, then groups = Lie groups.
- If $\mathcal{C} = \mathrm{VECT}$, then every object is a group in a unique way, with $m = +$.
- If $\mathcal{C} = \mathrm{ComRing}^{op} = \mathrm{AffSch}$, then groups = *affine algebraic groups*.

E.g. $\mathbb{G}_m = GL(1) = \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$ with multiplication

$$m_2 : \mathrm{Spec}(\mathbb{Z}[t,t^{-1}]) \times \mathrm{Spec}(\mathbb{Z}[t,t^{-1}]) \to \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$$

$$\mathrm{Spec}(\mathbb{Z}[t,t^{-1}]) \times \mathrm{Spec}(\mathbb{Z}[t,t^{-1}]) = \mathrm{Spec}(\mathbb{Z}[t_1,t_1^{-1},t_2,t_2^{-1}]) \xrightarrow{\mathrm{Spec}(t_1 t_2 \leftarrow\!\shortmid t)} \mathrm{Spec}(\mathbb{Z}[t,t^{-1}]).$$

Note: $\mathrm{Spec}(R) \times \mathrm{Spec}(S) = \mathrm{Spec}(R \otimes S)$ because we work with *commutative* rings. Coproduct of non-com rings is much larger. The diagonal map

$$\mathrm{Spec}(R)^{\times 1} \to \mathrm{Spec}(R)^{\times 2}$$

is spec(multiplication $R \otimes R \to R$), which is only a ring homomorphism if R is commutative.

<u>My favorite example</u>: Universal enveloping algebras are groups in $\mathcal{C}$ = Cocommutative Coalgebras.

<u>Defn:</u> If $\mathfrak{g} \in \mathrm{Vect}$ is a Lie algebra, $\mathcal{U}\mathfrak{g} \in \mathrm{Alg}$ is the universal associative algebra s.t.

$$\mathrm{Hom}_{\text{algebras}}(\mathcal{U}\mathfrak{g} \to A) = \mathrm{Hom}_{\text{Lie algebras}}(\mathfrak{g} \to A \text{ w/ commutator bracket}).$$

It is constructed from the free ("tensor") algebra:

$$\mathcal{U}\mathfrak{g} = \frac{\bigoplus \mathfrak{g}^{\otimes n}}{\substack{\text{ideal generated by} \\ [x,y] - x\otimes y + y \otimes x \\ \text{for } x,y \in \mathfrak{g}}}.$$

<u>Lemma:</u> $\mathcal{U}$ is a functor taking $\oplus \mapsto \otimes$. Proof: obvious that $\amalg \mapsto \amalg$. $\oplus, \otimes$ are "$\amalg$ / two pieces commute".

<u>Cor:</u> There is a diagonal map $\mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$, $x \longmapsto (x,x)$. It determines

$$\mathcal{U}\mathfrak{g} \xrightarrow{\Delta} \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$$

$\underset{\mathfrak{g}}{\overset{x}{\wedge}} \longmapsto x \otimes 1 + 1 \otimes x$; extend as algebra hom.

It makes $\mathcal{U}\mathfrak{g}$ into a monoid in CocomCog.

<u>Lemma:</u> $\underset{\mathfrak{g}}{\overset{x}{\wedge}} \mapsto -x$ extends as alg anti-hom to a $\star$-inverse for $\mathrm{id}|_{\mathcal{U}\mathfrak{g}}$, so $\mathcal{U}\mathfrak{g}$ is a <u>group</u>.

Another example:

X = any set. $\mathbb{K}X$ = vector space with basis X. $\mathbb{K}\cdot$ is a functor taking $\times \mapsto \otimes$. So diagonal map $X \to X \times X$ induces $\Delta : \mathbb{K}X \to \mathbb{K}X \otimes \mathbb{K}X$, $x \mapsto x \otimes x$.

So $\mathbb{K}X$ is a cocommutative coalgebra.

If $X = G$ is a discrete group, then $\mathbb{K}G$ is a group in CocomCog.

Geometric Aside:

By PBW theorem, $U\mathfrak{g} \cong \mathrm{Sym}(\mathfrak{g})$ as cocom coalgebras. (In char $= 0$, an explicit iso is available.) You should think of $\mathrm{Sym}(\mathfrak{g})$ = "infinitesimal nbhd of $0 \in \mathfrak{g}$," e.g. because algebra dual $(\mathrm{Sym}(\mathfrak{g}))^* =$ Formal power series in variables $\mathfrak{g}^*$, at least in char $= 0$. Also, there is a unique map of coalgebras $\mathbb{K} \to \mathrm{Sym}(\mathfrak{g})$, i.e. $\mathrm{Sym}\,\mathfrak{g}$ has a unique "point" inside it. So $U\mathfrak{g}$ is an "infinitesimal" or "formal" group with Lie algebra $\mathfrak{g}$.

§ Generalization:

In commutative geometry, the diagonal map $X \to X \times X$ is canonical. In noncommutative geometry, it is extra structure.

Note: in any category, if X is ~~a monoid~~ an algebra and Y is a coalgebra, then $\mathrm{Hom}(Y, X)$ ~~is~~ is a monoid.

Defn: Let $\mathcal{C}$ be a category w/ symmetric $\otimes$.

A *hopf algebra* is an object $X \in \mathcal{C}$ with:

- an ~~monoid~~ algebra structure  st. ...
- a coalgebra structure  st ...
- such that the coalgebra ~~maps~~ maps are homomorphisms for the algebra structure, or equivalently the algebra maps are homomorphisms for the coalgebra structure
- and $id \in \mathrm{Hom}(X,X)$ is $*$-invertible.

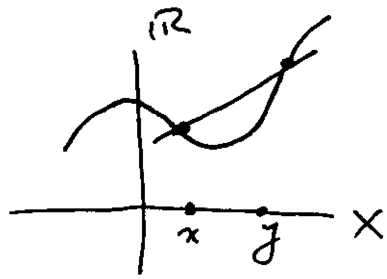
Example: Universal enveloping algebras + Affine algebraic gps both give hopf algebras in VECT.

§ C[G] vs. $U\mathfrak{g}$

Let G be a Lie (or algebraic) group with Lie algebra $\mathfrak{g}$. We said that $U\mathfrak{g}$ "is" ~~roughly~~ the "subgroup" of G consisting of "points infinitesimally close to the identity". Better intuition: $U\mathfrak{g}$ = "linear combinations of points infinitesimally close to $e \in G$".

Of course there is only one such point. But there are linear combinations. For example, if $x, y \in X$ ~~$\mathbb{R}$~~, then the formal linear combination $x \ominus y$ measures the "slope" of a "secant line".

I.e.: if $f : X \to \mathbb{R}$ any function, extend to $f : \mathbb{R}X \to \mathbb{R}$ by linearity. Then $f(x - y) = f(x) - f(y)$



Now let ~~y~~ $x \to$ ~~y~~ x along some curve ~~y~~ $x = x(t)$, ~~y~~ $= x(0)$.

Consider $\frac{1}{t}\left(x(t) - x(0)\right)$. As $t \mapsto 0$, this computes

$$f \longmapsto \frac{\partial f}{\partial x} \cdot \dot{x}(0).$$

This motivates:

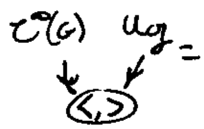
Theorem: There is a ~~can~~ canonical pairing $C^\infty(G) \otimes U\mathfrak{g} \to \mathbb{R}$ in which $1 \in U\mathfrak{g}$ is "evaluate at $e \in G$" and $x \in \mathfrak{g}$ is "differentiate f in the x direction" using $\mathfrak{g} = T_e G$.

<u>Pf:</u> Since right-multiplication by $g \in G$ is an iso $G \to G$, the tangent bundle TG is trivializable by

$$T_g G \overset{\sim}{\longleftrightarrow} T_e G : d(\cdot g)$$

Thus $\mathfrak{g} \hookrightarrow$ left-invariant vector fields = some derivations. (In fact, this is an iso). This is a Lie algebra map (or maybe off by left $\leftrightarrow$ right).

Thus we get $U\mathfrak{g} \longrightarrow$ Differential Operators on G an algebra homomorphism. [illegible] We set



$$\langle , \rangle : C^\infty(G) \otimes U\mathfrak{g} \longrightarrow \mathbb{R}$$

$$(f, a) \longmapsto \left(\begin{matrix}\text{a extended to}\\ \text{a left-inv}\\ \text{diff op}\end{matrix}\right)[f]\Big|_e . \quad \square$$

(or use diff ops supported at identity: $\mathcal{D}_G = \{\alpha : 1KG \to k$ s.t. $\alpha(I_e^N) = 0$ for $N \gg 0\}$, I_e = ideal of e.)

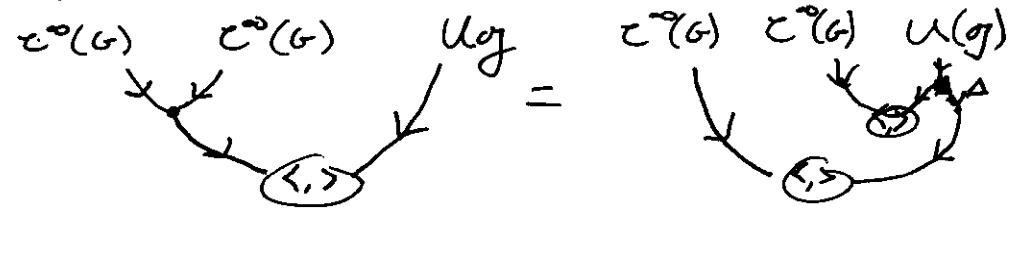
<u>Note:</u> $C^\infty(G)$ is a "topological Hopf algebra" because

$$m : G \times G \to G \rightsquigarrow C^\infty(G) \xrightarrow{\Delta} C^\infty(G \times G)$$

$$\circlearrowleft \text{Dense} \quad C^\infty(G) \otimes C^\infty(G)$$

It is a group in <u>Frechèt</u> or <u>Nuclear</u> [illegible]^{op} [illegible] $\mathrm{CommAlg}^{op}$.

<u>Theorem:</u> The pairing $\langle , \rangle$ is a "Hopf Pairing". I.e. [illegible]



and reverse. (Part of the claim is that if $a, b \in U\mathfrak{g}$, then ~~[struck out]~~

$$\langle -, b\rangle \otimes \langle -, a\rangle : \mathcal{C}^\infty(G) \otimes \mathcal{C}^\infty(G) \to \mathbb{R}$$

actually converges on the completion $\mathcal{C}^\infty(G \times G)$.)

Reason: Commutative multiplication on $\mathcal{C}^\infty(G)$ and cocommutative comultiplication on $U\mathfrak{g}$ both encode the "geometry" of G (and its infinitesimal nbhd of $e \in G$).

Comult on $\mathcal{C}^\infty(G)$ and mult on $U\mathfrak{g}$ both encode the group structure of G.

Compare:
Replace G by a discrete group, $U\mathfrak{g} \rightsquigarrow \mathbb{K}.H$, $\mathcal{C}^\infty(G) \rightsquigarrow$ Functions$(G \to \mathbb{K})$.
for H some subgroup of G,

Theorem: ~~Re~~ Assume G is connected and replace $\mathcal{C}^\infty(G)$ by ~~A~~ Real-analytic functions; or G = algebraic and $\mathcal{C}^\infty \rightsquigarrow$ Poly.
Then the pairing $\mathcal{O}(G) \otimes U\mathfrak{g} \to \mathbb{R}$ is perfect:
the two maps $\mathcal{O}(G) \to (U\mathfrak{g})^*$ and $U\mathfrak{g} \to (\mathcal{O}(G))^*$ are injective.

Pf: $U\mathfrak{g} \to (\mathcal{O}(G))^*$ is always injective: a diff op is determined by how it acts on polynomials. That $\mathcal{O}(G) \to (U\mathfrak{g})^*$ is injective is Taylor's theorem that an analytic function is determined by its Taylor expansion.

§ Upper triangular matrices.

The problem is that $\mathcal{U}\mathfrak{g}$, $\mathcal{O}(G)$ are usually infinite-dimensional, so we cannot say "$\mathcal{O}(G) = (\mathcal{U}\mathfrak{g})^*$", or anything like that — this is why we talked instead about "perfect pairings". But there is one case when you can take duals of ∞-dim spaces: if they're graded, and each piece is finite-dim. This happens for upper-triangular matrices. I will do $\mathfrak{sl}(n)$ case, but of course the discussion generalizes to any semisimple Lie algebra over $\mathbb{C}$.

<u>Defn:</u> $\mathfrak{n} = \mathfrak{n}_+ =$ Lie alg of upper triangular matrices

$$= \begin{pmatrix} 0 & \ddots & * \\ 0 & & 0 \end{pmatrix}.$$

It has basis E_{ij}, $1 \le i < j \le n$, with bracket

$$[E_{ij}, E_{jk}] = E_{ik}, \quad [E_{ij}, E_{k\ell}] = 0 \text{ if } \begin{smallmatrix} j \ne k \\ i \ne \ell \end{smallmatrix}.$$

So it is graded by $|E_{ij}| = j - i$ (i.e. $[,]$ preserves grading). It is generated by $E_{i,i+1}$ for $1 \le i < n$.

Then $\mathcal{U}\mathfrak{n}$ is also $\mathbb{N}$-graded, generated by its degree-1 piece, and so each graded component is finite-dim:

$$\mathcal{U}\mathfrak{n} = \bigoplus_{\ell \ge 0} (\mathcal{U}\mathfrak{n})_\ell$$

where $(U\mathfrak{n})_0 = \mathbb{K} \hookrightarrow U\mathfrak{n}$, $((U\mathfrak{n}))_\ell$ = linear combos of products of ℓ $E_{i,i+1}$ s.

In particular, $\dim((U\mathfrak{n})_\ell \leq (n-1)^\ell$.

Set $(U\mathfrak{n})^*_{gr} = \bigoplus_{\ell \geq 0} ((U\mathfrak{n})_\ell)^*$. It is a commutative Hopf algebra, and so $= \mathcal{O}(G)$ for some affine alg gp $G = \mathrm{Spec}((U\mathfrak{n})^*_{gr})$.

Conversely, set N = gp of upper triangular matrices

$$= \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

It is an algebraic group: as spaces, $N \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathfrak{n}$, and

since $\mathfrak{n}$ = nilpotent, BCH formula truncates to define the multiplication on N: $x \cdot y = x + y + \frac{1}{2}[x,y] + \dots$, $x,y \in \mathfrak{n}$, which is a polynomial, so defines

$$\mathrm{BCH}: \mathcal{O}(N) \to \mathcal{O}(N) \otimes \mathcal{O}(N)$$

$\mathcal{O}(N)$ = Polynomial functions on $\mathfrak{n}$ $= \mathrm{Sym}(\mathfrak{n}^*)$.

But $\mathfrak{n}^*$ is graded (via $\mathfrak{n}$'s grading), so $\mathcal{O}(N) = \mathrm{Sym}(\mathfrak{n}^*)$ is graded, again with f.dim pieces. Our earlier discussion built a perfect pairing

$$U\mathfrak{n} \otimes \mathcal{O}(N) \to \mathbb{R}$$

and this pairing respects the gradings. By local-finite-dimensionality, we do have

$$(U\mathfrak{n})^*_{gr} = \mathcal{O}(N)$$

as graded Hopf algebras: the map $\mathcal{O}(N) \hookrightarrow (U\mathfrak{n})^*_{gr}$ is an iso by loc.f.dim, just like a perfect pairing between finite-dim v-spaces makes each the other's dual.