Introduction to Q-manifolds and BRST

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11 October 2011
"Witten in the 80s" student seminar

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The primary reference for all things graded- and Q-manifold related is the 2006 PhD thesis by Rajan Mehta. Most of this material is (after Raj's thesis) well-known; the description of Faddeed-Popov construction for Lie algebroids is joint work with Dan Berwick-Evans.

"BRST" stand respectively for Becchi, Rouet, and Stora, who published some joint papers in the US explaining why FP's construction (which was originally geometrically motivated) gives results that don't depend on choices, and basically introduced cohomological arguments to physics; and Tyutin, who came up with the same results but was stuck behind the Iron Curtain.

1 Definition of Q-manifold

Suppose I have a space X with a group action $G \curvearrowright X$. Recall the *GIT quotient*: $[X/G] = \operatorname{spec}(\hom_G(1, \mathscr{C}^\infty(X)))$. Here 1 denotes the trivial G-module, so that $\hom_G(1, -)$ is the functor of invariants. $[X/G]$ is (more or less) the *coarse quotient* or *space quotient* of X; its points are the orbits of the G-action on X, or rather the closures thereof.

Since we are in the 21st century, we know that more detailed information is available: you could take the *derived functor* of invariants. Note that $\mathrm{Rhom}_G(1, \mathscr{C}^\infty(X))$ (the chain complex that computes $\mathrm{Tor}^\bullet_G(1, \mathscr{C}^\infty(X)))$ is defined only up to homotopy, and is a "strongly-homotopy commutative algebra." Let's pretend that it is actually a commutative dga — I don't want to think to hard about "E_∞ ring spectra" or whatever is now in vogue. Then "spec" of commtuative dgas isn't too bad, and I'd call

$$X//G \overset{\text{def}}{=} \operatorname{spec}(\mathrm{Rhom}_G(1, \mathscr{C}^\infty(X))).$$

One place it is likely to be an actual (and not-too-large) commutative (dg) algebra is if the "group" acting on X is actually a Lie algebra.

Remark: Rhom doesn't record much about finite group actions, because we are in characteristic-zero. So let's just work with infinitesimal actions. ♢

Example: If $\mathfrak{g}$ is a Lie algebra, what is $\{\mathrm{pt}\}//\mathfrak{g}$? Well, what is $\mathrm{Rhom}(1,1)$? We resolve:

$$\cdots \to \mathcal{U}\mathfrak{g} \otimes \mathfrak{g}^{\wedge 2} \to \mathcal{U}\mathfrak{g} \otimes \mathfrak{g} \to \mathcal{U}\mathfrak{g} \to 1$$

Then we take $\hom_{\mathfrak{g}}$ from this to the trivial module. Well, $\hom_{\mathfrak{g}}$ from a free module is the space of linear maps from the generating vector space. So we see that

$$\mathrm{Rhom}_{\mathfrak{g}}(1,1) = \hom_{\mathbb{K}}(\textstyle\bigwedge^\bullet \mathfrak{g}, 1) \text{ with some differential} \overset{\text{def}}{=} \mathrm{CE}^\bullet(\mathfrak{g}).$$

More generally, for any $\mathfrak{g}$-module M, $\mathrm{Rhom}_{\mathfrak{g}}(1,M) \overset{\text{def}}{=} \mathrm{CE}^\bullet(\mathfrak{g};M) = \bigwedge^\bullet(\mathfrak{g}^*) \otimes M$ as a graded vector space. (The differential is "twisted" and not the tensor-product differential, and encodes the adjoint action of $\mathfrak{g}$ on itself and the action of $\mathfrak{g}$ on M.) Anyway, this is true if $\dim \mathfrak{g} < M$, which for now we assume is true.

So as an algebra, $\mathrm{Rhom}(1,1) = \bigwedge^\bullet(\mathfrak{g}^*)$. Then $\{\mathrm{pt}\}//\mathfrak{g} = \mathrm{spec}(\bigwedge^\bullet(\mathfrak{g}^*)) = \pi\mathfrak{g}$, but it has some extra structure (the differential). ♢

More generally, $X//\mathfrak{g} = \mathrm{spec}(\mathscr{C}^\infty(X) \otimes \mathrm{CE}^*(\mathfrak{g})) = X \times \pi\mathfrak{g}$, except it has more structure than this. The $\mathbb{Z}$-grading records the rescaling action on $(\mathbb{R}, \times)$ on $\pi\mathfrak{g}$.

Notation: *I will write π(vector bundle) when I want you to think of the $\mathbb{Z}$-grading as in the above example. I will write π^{-1} for the other-direction shift. So $\pi = \pi^{-1}$ as supermanifolds, but they have different inherent $\mathbb{R}$-actions.*

Example: There are other times when you have a good notion of "invariant function." For example, if X comes equipped with an integrable (smooth, constant rank) distribution $D \subseteq \mathrm{T}X$, then the GIT quotient $[X/D]$ is spec(D-invariant functions on X). We can define $X//D$ by doing the derived thing.

Then $X//D = \pi D$ with: $\mathbb{Z}$-grading as a vector bundle, and some differential.

For example, $X//\mathrm{T}X = \mathrm{spec}(\Omega^\bullet X)$, with the differential = the de Rham d. This might be a good definition of de Rham complex. ♢

Notation: $X//\mathrm{T}X \overset{\text{def}}{=} X_{\mathrm{dR}}$.

Remark: There is a common generalization of these two examples (distributions, Lie algebra actions), called *Lie algebroid*. It is a (classical) vector bundle $A \to X$ and some structure that you can look up encoding the idea that "A acts infinitesimally on X". Then $X//A = \pi A$ with some differential. ♢

These are all (spec of) dgas, which is to say the differential is a *derivation*, which you should think of as a *vector field*. The $\mathbb{Z}$-grading is also a derivation $f \mapsto nf$ if f is homogeneous with $|f| = n \in \mathbb{Z}$.

Definition: *A* Q-manifold *is a supermanifold with:*

- *a $\mathbb{Z}$-grading, i.e. a (bosonic) vector field* $\mathbf{n}$ *(the "ghost number operator")*
- *a "Q-structure", i.e. a* fermionic *vector field Q such that:*

$$[Q,Q] = 0 \qquad \textit{non-trivial for fermionic vector fields}$$
$$[\mathbf{n},Q] = +1 \cdot Q \qquad \textit{cohomological grading}$$

(Actually, homological grading is more geometric, but cohomological grading is too entrenched.) As I will not need more generality, I will ask that $\mathbf{n}$ *agrees* $\bmod 2$ *with the fermion count. It's usually requested also that the manifold have local charts with all coordinates homogeneous for* $\mathbf{n}$*.*

What is a *morphism* of Q-manifolds? The algebraic answer: a morphism of dgas in the other direction. The geometric answer is better. Just think classically. If Ξ, Υ are manifolds with vector fields $\mathbf{v}_\Xi, \mathbf{v}_\Upsilon$ (let's call a manifold equipped with a vector field a *V-manifold*), then any smooth map $f : \Xi \to \Upsilon$ determines: pulled-back vector bundle $f^*\mathrm{T}R \to S$; map $\mathrm{d}f : \mathrm{T}R \to f^*\mathrm{T}S$ of vector bundles over R. Then f is a *V-morphism* if $\mathrm{d}f \cdot \mathbf{v}_\Xi = f^*\mathbf{v}_\Upsilon$, i.e. if $\mathrm{d}f \cdot \mathbf{v}_\xi = \mathbf{v}_{f(\xi)}$ for each $\xi \in \Xi$.

We can now construct the *inner hom* of V-manifolds. If Ξ, Υ are V-manifolds, then first build $\mathrm{Maps}(\Xi, \Upsilon)$, the (infinite-dimensional) manifold of all smooth maps $\Xi \to \Upsilon$. Note that $f^*\mathbf{v}_\Upsilon - \mathrm{d}f \cdot \mathbf{v}_\Xi$ measures the failure of a smooth map to be a V-map. Note also that the *global points* (i.e. morphisms from $\{\mathrm{pt}\}$) of a V-manifold $(\Xi, \mathbf{v}_\Xi)$ are precisely the vanishing locus of $\mathbf{v}_\Xi$. So we want to give $\mathrm{Maps}(\Xi, \Upsilon)$ a vector field whose zeros are those f for which $f^*\mathbf{v}_\Upsilon - \mathrm{d}f \cdot \mathbf{v}_\Xi = 0$. Well, a little thought shows that if $f : \Xi \to \Upsilon$ is smooth, then $\mathrm{T}_f \mathrm{Maps}(\Xi, \Upsilon) = \Gamma_\Xi(f^*\mathrm{T}\Upsilon)$. So $\mathbf{v}_\Upsilon$ and $\mathbf{v}_\Xi$ each define a tangent vector at f, and we can define on $\mathrm{Maps}(\Xi, \Upsilon)$ the vector field whose value at f is:

$$\mathbf{v}_{\mathrm{Maps}}|_f = f^*\mathbf{v}_\Upsilon - \mathrm{d}f \cdot \mathbf{v}_\Xi$$

I.e. $f^*\mathbf{v}_\Upsilon - \mathrm{d}f \cdot \mathbf{v}_\Xi$ is a vector field, and is the vector field we want.

Remark: The sign is the same as in the inner hom of $\mathfrak{g}$-modules. $\Diamond$

Exercise: *Show that this V-structure makes* $\mathrm{Maps}(\Xi, \Upsilon)$ *into the inner hom in the category of V-manifolds. I.e. prove the hom-tensor adjunction.*

Similarly we get an inner hom in the category of Q-manifolds.

Exercise: *You can, of course, forget Q and remember just the $\mathbb{Z}$-grading* $\mathbf{n}$*. Show that $X \mapsto X_{\mathrm{dR}}$ is right-adjoint to this forgetful functor.*

Exercise: *Let ϵ denote the standard (odd) coordinate on $\pi^{-1}\mathbb{R}$. You can give $\pi^{-1}\mathbb{R}$ a Q-structure by setting $Q = \frac{\partial}{\partial \epsilon}$.*

If X is a graded manifold, you can make it into a Q-manifold by setting $Q = 0$; I will abuse notation and call the corresponding Q-manifold also by X. Show that

$$X_{\mathrm{dR}} = \mathrm{Maps}(\pi^{-1}\mathbb{R}, X)$$

(inner hom of Q-manifolds; natural in X)

Remark: X_{dR} is not a Q-vector bundle, because $X_{\mathrm{dR}} \to X$ is not a Q-morphism. But $\pi \mathrm{T}X$ is a vector bundle over X in the category of graded or super manifolds. A question I don't know the answer to: how natural is this vector bundle structure? I.e.: there is an addition map $+ : \pi\mathrm{T}X \times_X \pi\mathrm{T}X \to \pi\mathrm{T}X$. If this were natural in X, then since $\pi\mathrm{T} = \mathrm{Maps}(\pi^{-1}\mathbb{R}, -)$ in graded manifolds, we should have a map $\pi^{-1}\mathbb{R} \to \pi^{-1}\mathbb{R} \cup_{\{\mathrm{pt}\}} \pi^{-1}\mathbb{R}$. But this colimit doesn't exist in manifolds, and in "spaces" there is no such "diagonal" map. ◊

Definition: *If (Ξ, Q) is a Q-manifold, the* cohomology $\mathrm{H}^\bullet(\Xi)$ *is the cohomology of the dga $(\mathscr{C}^\infty(\Xi), Q)$. E.g.* $\mathrm{H}^\bullet(X_{\mathrm{dR}}) = \mathrm{H}^\bullet_{\mathrm{dR}}(X)$.

Definition: *Some types of morphisms:*

- $f : \Xi \to \Upsilon$ *is a* quasi-isomorphism *if it is an iso on homology.*
- $f : \Xi \leftrightarrow \Upsilon : g$ *are* quasi-inverse *if $f \circ g$ and $g \circ f$ are isos on homology.*

Remark: Having a quasi-inverse is *strictly stronger* than being quasi-iso. E.g. $\emptyset \to (\pi^{-1}\mathbb{R}, \frac{\partial}{\partial \epsilon})$. In Quillen–Sullivan "rational homotopy theory," it is requested that the dgas have no support in negative cohomological degree. Then I think quasi-iso and quasi-invertible are the same, but I'm not sure? ◊

Notation: *In physics, Q is called the* BRST operator. *In math it is often called the* cohomological vector field. *Why "Q"? It is the first letter of "cohomological."*

2 Towards a Q-manifold integration theory: the BRST trick

We'd like to compute "path" integrals. As a warm-up, we study finite-dimensional integrals. What if the "space" of "paths" is a stack X/G (with some meaning of "stack") for G a compact group and X a space? Then the "physical observables" should be G-invariant functions on X, i.e. $\mathrm{H}^0(X//G)$. What is a good type of measure? If X has a G-invariant measure μ_X, we could integrate $\int_X f\, \mu_X$, where $f = e^s$ for S some *action*. But the dimensions are wrong: we want something with scaling dimension $\dim X - \dim G$. Anyway, we're clearly overcounting: if $G = \mathbb{Z}/2$, then $\mathrm{Vol}(X) = 2\,\mathrm{Vol}(X/G)$. So we should pick a Haar measure μ_G, and we expect $\int_{X/G} = \frac{1}{\mathrm{Vol}(G)} \int_X$. So a "measure" on X/G is a kind of ratio: a measure on X divided by a measure on G.

Note that a Haar measure on G is the same data, when $\mathrm{Lie}\, G = \mathfrak{g}$, as an ad-invariant measure on $\mathfrak{g}$. If we feel safe in ignoring the discrete data in G $(\pi_0, \pi_1, \dots)$ then we feel safe in modeling X/G as the derived quotient $X//\mathfrak{g}$, which is a Q-manifold with underlying supermanifold $X \times \pi\mathfrak{g}$. You can check that μ_X/μ_G transforms as a section of the Berezinian line over $X \times \pi\mathfrak{g}$. But it is defective, in two ways:

1. $[\mathbf{n}, \frac{\mu_X}{\mu_G}] = [\mathbf{n}, \mu_X]\frac{1}{\mu_G} - [\mathbf{n}, \mu_G]\mu_X = (0 - \dim \mathfrak{g})\frac{\mu_X}{\mu_G}$, not 0. So integrals of cohomological-degree-0 functions against this measure are identically 0. E.g. if X is oriented, then $\pi\mathrm{T}X$ has

a canonical Berezinian measure, which is the "integrate top forms" measure. But we want to integrate functions, not forms.

2. If $s \in \mathscr{C}^\infty(X)$ is $\mathfrak{g}$-invariant, then it does give a function on $X//\mathfrak{g} = X \times \pi\mathfrak{g}$. But this function has a huge critical locus (it is constant on $\mathfrak{g}$-orbits in X and on fibers of $X//\mathfrak{g} \to X$ — this is a graded manifold map, not a Q-manifold map), and we normally want to study $\int e^s$ by stationary phase, etc.

I'll explain how to solve 2. first. This is the *BRST trick*.

Suppose that we have a Q-manifold Υ with $\mathscr{C}^\infty(\Upsilon)$ including functions of *negative* degree. Also, we have:

- an action s, which is closed and in cohomological-degree 0 (equivalently, $s : \Upsilon \to \mathbb{R}$ is a morphism of Q-manifolds) — we want to compute $\int_\Upsilon \exp(s)$.
- a Berezinian measure $\mu = \mu_\Upsilon$ in cohomological degree 0. We should demand some "Q-ness" of the measure; the correct demand is that $\mathcal{L}_Q \mu_\Upsilon = 0$, where $\mathcal{L}_{\mathbf{v}}$ is the Lie derivative in the $\mathbf{v}$ direction.
- some choice of $f \in \mathscr{C}^\infty(\Upsilon)$ which is in cohomological degree -1.

Then consider, for $t \in [0, 1]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Upsilon \exp(s + tQ[f])\,\mu &= \int_\Upsilon \frac{\mathrm{d}}{\mathrm{d}t}\exp(s + tQ[f])\,\mu \\ &= \int_\Upsilon \exp(s + tQ[f])\,Q[f]\,\mu \\ &= \int_\Upsilon \mathcal{L}_Q\big(\exp(s + tQ[f])\,f\,\mu\big) \\ &= 0\end{aligned}$$

by Stoke's formula. The last line holds only if there is sufficient convergence to assure that the "boundary" of Υ does not contribute to the integral.

It follows that:

$$\int \exp(s) = \int \exp(s + Q[f])$$

This is not too much of a surprise. The idea is that the "physical observables" should be just $\mathrm{H}^0(\Upsilon)$, and not all closed functions — or at least that an integral should really only depend on the cohomology class of the function.

The point is: even if s has a large critical locus, perhaps $s + Q[f]$ doesn't.

So to sum up, the usual situation is that we have a Q-manifold Ξ and a function $s : \Xi \to \mathbb{R}$ (closed of cohomological degree 0) with too large of a critical locus, and some data that really ought to make a measure on X but gives something in the wrong cohomological degree, and we are looking for: a Q-manifold Υ and

- a quasi-isomorphism $\Upsilon \overset{\sim}{\to} \Xi$. Then we can pull back $s \in \mathscr{C}^\infty(\Xi)$ to a function over Υ, and because the two manifolds are quasi-isomorphic we can hope that their coarse invariants (integrals of functions, etc.) don't change.
- Υ should have enough functions in cohomological degree -1 to be able to find something cohomologous to f with a small enough critical locus. (Υ should not have *so* many functions as to be unwieldy.)
- Υ should have a good choice of measure in degree 0.

3 The Faddeev-Popov construction

To build such an Υ, let us note that in most cases of physical interest the Q-manifold Ξ is a derived quotient $\Xi = X//A$ for some Lie algebroid A, i.e. $\Xi = \pi A$ with some Q-structure. (More interesting things can also happen. For example, there might be "symmetries among symmetries", and then Ξ will have coordinate functions also in cohomological-degree $+2$, etc.) In particular, there's a map $\Xi \to X$ of graded manifolds. We will use this to build Υ.

Definition: *Suppose that $f : Y \to X$ is a submersion of graded manifolds, Ξ is a Q-manifold, and $\Xi \to X$ is a map of* graded *manifolds. The* Q-pullback *(aka "Lie algebroid pullback" or "double pullback") of Ξ along f is the pullback in Q-manifolds:*

$$\begin{array}{ccc} \Upsilon & \xrightarrow{\tilde f} & \Xi \\ \big\downarrow & \lrcorner & \big\downarrow \\ Y_{\mathrm{dR}} & \xrightarrow{\pi \mathrm{T} f} & X_{\mathrm{dR}} \end{array}$$

(Recall that $\Xi \to X$ of graded manifolds induces $\Xi \to X_{\mathrm{dR}}$ of Q-manifolds.) The pullback exists because if $Y \to X$ is a submersion of graded manifolds, then $Y_{\mathrm{dR}} \to X_{\mathrm{dR}}$ is a submersion of Q-manifolds.

Remark: Locally, we can write $Y = X \times F$ for some fiber F. Then locally $Y_{\mathrm{dR}} = X_{\mathrm{dR}} \times F_{\mathrm{dR}}$; so locally $\Upsilon = \Xi \times F_{\mathrm{dR}}$. ◊

Theorem (—, DBE): *If $Y \to X$ is a (graded) vector bundle, then $\Upsilon \to \Xi$ is quasi-invertible.*

Remark: This result shouldn't be too much of a surprise. You know that if $Y \to X$ is a vector bundle, then $Y_{\mathrm{dR}} \to X_{\mathrm{dR}}$ has quasi-inverse the zero section; the result says that this generalizes to Q-pullbacks. The intuition is the local description of $\Upsilon = \Xi \times F_{\mathrm{dR}}$; if the fiber F of $Y \to X$ is a vector space, then each fiber F_{dR} of $\Upsilon \to \Xi$ is equivalent to $\{\mathrm{pt}\}$.

The proof does not require addition, just that a vector space is "star-shaped" — it comes with a distinguished homotopy to the origin. The proof is pretty much completely algebraic, and so applies in the infinite-dimensional case as well (provided infinite-dimensional Q-manifolds are made precise). $\Diamond$

Now we can essentially solve the problem from the previous section. In the standard examples, $\Xi = X//A$ comes equipped with a Berezinian measure which seems to encode the correct data but is in negative cohomological degree (it is non-zero on "top forms"). Suppose that the vector bundle $Y \to X$ is "oriented" in the sense that each fiber is given an orientation, which is constant as you move around X. Then each fiber F_{dR} of $\Upsilon \to \Xi$ receives its canonical "integrate top forms" measure. By controlling the graded dimension of F, we can make the product measure on $\Upsilon = \Xi \times F_{\mathrm{dR}}$ have cohomological degree 0.

Definition (Faddeev-Popov complex): *Let $\Xi = X//A$ for $A \to X$ a Lie algebroid, so that $\Xi = \pi A$ as a graded manifold. Then to get the cohomological degrees to work out, it suffices to take $Y = \pi^{-1}B$ where $B \to X$ is a classical vector bundle with the same rank as $A \to X$.*

When A is finite-dimensional (and oriented), the Faddeev-Popov choice *is $B = A^*$ the dual vector bundle, so that $Y = \pi^{-1}A^* = (\pi A)^*$.*

Remark: When $A \to X$ is not oriented, then you should make some twists. Over $\mathbb{R}$, this is easy. Let X be a manifold, and $L \to X$ a line bundle. Then $L^{\otimes 2}$ comes equipped with an orientation, because all squares are positive, and so $|L| = \sqrt{L^{\otimes 2}}$ is well-defined as the unique oriented line bundle that squares to $L^{\otimes 2}$. An *orientation* of L is a trivialization of the $\mathbb{Z}/2$-bundle $L/|L|$. An orientation of a vector bundle is an orientation of its top exterior power. If A is not oriented, then the correct choice is to twist A^* by $A^{\wedge \mathrm{top}}/|A^{\wedge \mathrm{top}}|$. $\Diamond$

Remark: Non-canonically (it amounts essentially to a choice of connection), we can write $\Upsilon = (\pi^{-1}A^*)_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} \Xi \cong \pi^{-1}A^* \times_X A^* \times_X \pi A$. When $A \to X$ is trivialized $A = X \times \mathfrak{g}$ for a Lie algebra action, then there is the trivial connection, and $\Upsilon = X \times \pi^{-1}\mathfrak{g}^* \times \mathfrak{g}^* \times \pi\mathfrak{g}$. In physics, these factors are sometimes called "fields", "anti-ghosts", "Lagrange multipliers", and "ghosts", respectively. $\Diamond$

Finally, how do we choose an exact function?

Definition (Faddeev-Popov gauge fixing): *Suppose that $A \to X$ is an oriented Lie algebroid, and construct Υ as above, so that Υ presents $X//A$ as a derived quotient. Choose $f : X \to A$ a section of $A \to X$. This choice amounts to a linear function on $\pi^{-1}A^*$ which is in cohomological degree -1. This function lifts to a (non-closed!) function on $(\pi^{-1}A^*)_{\mathrm{dR}}$ and hence to a function on Υ; call this non-closed cohomological-degree-(-1) element of $\mathscr{C}^\infty(\Upsilon)$ also by f. Given an "action" $s \in \mathscr{C}^\infty(X)$ that is A-invariant, the* Faddeev-Popov gauge-fixed action *is $s + Q[f]$.*

Exercise: *Suppose the zero locus of $f : X \to A$ intersects the orbits of $A \curvearrowright X$ transversely (and is an embedded submanifold). Then after integrating out all the fibers of $\int_\Upsilon \exp(Q[f])\mu$ to get some measure just on X, the resulting measure is of the form (δ-measure on the zero locus)$\times$(det Jacobi).*

Suppose that $s \in \mathscr{C}^\infty(X)$ is A-invariant, and that the critical locus of s consists of an isolated orbit of the $A \curvearrowright X$. Suppose that (at least near this critical orbit) the zero-locus of $f : X \to A$ intersects the A-orbits transversely. Then the critical locus of the gauge-fixed action $s + Q[f]$ is an isolated pt, namely the intersection of the critical orbit of s and the zero-locus of f.