Orbifolds of CFTs and Cohomology of Sporadic Groups

Berkeley, 14 April 2017. Theo Johnson-Freyd

Caveat: This is work in progress. Some is mathematically rigorous, some requires physics! Joint w/ David Treumann

I'd like to begin by ~~explaining~~ explaining the following:

Thm (–, DT)
The first fractional Pontryagin class of the 24-dimensional representation of Co_0 has order 24. It generates a direct summand of $H^4(Co_0, \mathbb{Z})$. The other summand is one of the groups 0, $\mathbb{Z}_3$, $\mathbb{Z}_4$, or $\mathbb{Z}_{12}$.

Let me explain these pieces. Suppose G is a finite group and $V: G \to O(n)$ a real representation. It is oriented if it factors through $SO(n)$. The obstruction for this is $w_1(V) \in H^1(G, \mathbb{Z}_2)$, ~~the~~ where $w_1(V) = $ ~~V~~ $\det \circ V : G \to \mathbb{Z}_2$. A spin structure is a lift $G \xrightarrow{V} O(n)$ to $\mathrm{Spin}(n)$. The obstruction for this is $w_2(V) \in H^2(G, \mathbb{Z}_2)$. $w_2(V) = V^* w_2$ where $w_2 \in H^2(BO(n), \mathbb{Z}_2)$. Then ~~you choose a~~ $H^4(B\mathrm{Spin}, \mathbb{Z}) = \mathbb{Z}$, where $p_1 \in H^4(BO, \mathbb{Z})$ pulls back to twice the generator. The first fractional Pontryagin class is $V^* \frac{p_1}{2}$.

The Leech lattice is the unique self-dual unimodular even lattice of rank 24. It has many constructions. Here is one, which focusses on the prime $p=2$. Take the Golay Code

This is ~~the~~ a lagrangian subspace $\mathbb{Z}_2^{12} \hookrightarrow \mathbb{Z}_2^{24}$.
You can build it as the span of the following basis:

$$\begin{array}{cc} & \overbrace{\qquad\qquad}^{12}\ \overbrace{\qquad\qquad}^{12} \\ 12 & \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & X \\ & & 1 & \end{array}\right] \end{array}$$

where $X_{ij} = 1$ if vertices i,j are not adjacent ~~connected~~ in the icosahedron, and 0 if they are.

Take lattice $(\text{root}\,A_1^{\cancel{24}})^{24}$. $\text{root}\,A_1 = \sqrt{2}\cdot\mathbb{Z} \hookrightarrow \mathbb{R}$.
Its dual is $(\text{weight}\,A_1)^{24}$. $\text{weight}\,A_1 = A_1^* = \frac{1}{\sqrt{2}}\mathbb{Z}$.
Look at

$$\begin{tikzcd} N \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_1^{*24} \arrow[d] \\ \mathbb{Z}_2^{12} \arrow[r, hook, "\text{Golay}"'] & \mathbb{Z}_2^{24} = A_1^{24}/A_1^{*24} \end{tikzcd}$$

N is a Niemeier lattice. Choose a lift θ in N of $(1,1,\ldots,1) \in \text{Golay}$. N has a sublattice of index 2, namely $\ker(\langle\theta, -\rangle \bmod 2)$. This sublattice has a unique other way to extend to an even unimodular lattice. That other lattice is the Leech lattice.

Leech is funny. Its smallest vectors have $\langle v,v\rangle = 4$. Even more: Leech is chiral: it is not rotation-equiv to its reflection.

The automorphism group $\text{Aut}(\text{Leech})$ is called Co_0; of order $2^{22}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23 \sim 10^{17}$.

Its center is $\{\pm 1\}$, and $Co_0/\{\pm 1\} =: Co_1$ is simple. $H^2(Co_0, \mathbb{Z}) = H^3(Co_0, \mathbb{Z}) = 0$.

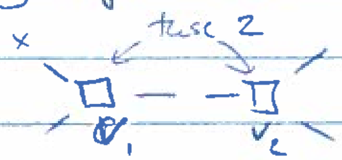
The automorphisms of Leech that preserve the index-2 sublattice Leech $\cap$ N is an important subgp of Co_0 of shape $2^{12} : M_{24} = \text{Golay} \rtimes M_{24}$, where $M_{24} \subseteq S_{24}$ is the automorphism gp of the Golay code. Many of the most important maximal subgps of Co_0 are (like this) stabilizers of Leech $\cap$ (other Niemeier lattice).

Let me tell you how we proved the theorem. First, the lower bound. Class 3B in M_{24} has cycle structure 3^8, and $\frac{p_1}{2}(\text{Leech}|_{\text{this } \mathbb{Z}_3})$ is non-zero in $H^4(\mathbb{Z}_3) = \mathbb{Z}_3$. So $o(\frac{p_1}{2})$ is divisible by 3.

Now, ask your computer to find a subgp of $2^{12} : M_{24}$ iso to $2D_{2\cdot 4}$, the binary dihedral gp of order 16, such that the central ± 1 in $2^{12} : M_{24}$ is the center of $2D_{2\cdot 4}$. (there are many) $2D_{2\cdot 4}$ is McKay gp E_6:

McKay gps have large H^4: $H^4(G \text{ mckay}) = \mathbb{Z}/|G|$ generated by $\frac{p_1}{2}$(underlying real rep of $\mathbb{C}^2$).

The only reps of $2D_{2\cdot 4}$ w/ central char -1 are



so ~~$\mathrm{Leech}\otimes\mathbb{R}|_{2D}$~~

$$\mathrm{Leech}\otimes\mathbb{R}|_{2D_{2\cdot 4}} = V_1^{\oplus a}\oplus V_2^{\oplus b}$$

where $a+b=6$, since $\dim_{\mathbb{R}} V_1 = \dim_{\mathbb{R}} V_2 = 4$.
But $\frac{p_1}{2}(V_1)=1$, $\frac{p_1}{2}(V_2)=9$, so

$$\frac{p_1}{2}\left(\mathrm{Leech}|_{2D_{2\cdot 4}}\right) = a + (6-a)9 = 54-8a$$

which is of order 8 in $\mathbb{Z}_6$ (In fact, $a=b=3$.)

For the upper bound, I recall a little more theory.
Suppose $G \geq S$. There are maps

$$H^\circ G \underset{\int}{\overset{\text{rest}}{\rightleftarrows}} H^\circ S$$

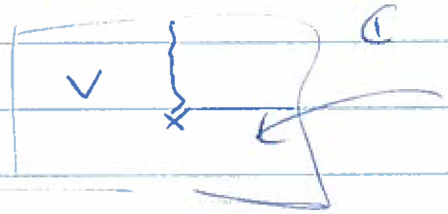
s.t. the composition is $\circlearrowleft$ = multiplication by $[G:S]$.
In particular, if $p \nmid [G:S]$, i.e. if S contains the p-Sylow in G, then $H^\circ G_{(p)}$ is a direct summand of $H^\circ S$.

The maximal subgps of Co_0 are all known.
E.g. $2^{12}:M_{24} \supseteq$ 2-Sylow. So go prime-by-prime.
Only $2^{12}:M_{24}$ and $3^6:2M_{12}$ are very interesting.
Constrain their H^4's w/ Serre spectral sequences.

So, why do I care?. Suppose V is a (holomorphic) chiral conformal field theory — a VOA or a conformal net or something. Then there is a distinguished class in

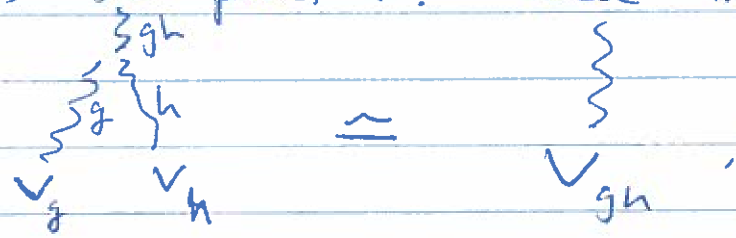
$$\alpha \in H^4(BAut\, V, \mathbb{Z})$$

called the <u>orbifold anomaly</u>. It has various constructions. Here is one. For each $g \in Aut\, V$, there is a <u>twisted sector</u>, which is the unique (up to iso) irreducible object in the category of ways to extend the field theory



V everywhere, monodromy g on the branch cut

across the puncture. These irreps can <u>fuse</u>:



and there is some element

$$\alpha(g,h,k) \in U(1)$$

which is the iso

$$V_{ghk} \simeq (V_g \boxtimes V_h) \boxtimes V_k \simeq V_g \boxtimes (V_h \boxtimes V_k) \simeq V_{ghk}$$

determined up to coboundaries.

<u>Warning</u>: these statements are known in CN world but not VOA world in general, so far as I can tell.

~~[scribble]~~ It is a little more complicated for super CFTs (not susy — I just mean there are fermions). Then the categories in question are supercategories: An irrep can be ordinary ($\cancel{\text{not}}$ $\mathrm{End} \cong \mathbb{C}$) or Majorana ($\mathrm{End} \cong \mathrm{Cliff}(1)$); isomorphisms can be even or odd. Then the anomaly is not really a class in ordinary cohomology but in supercohomology...

$$\to H^4_{GW}(G) \longrightarrow H^4_{super}(G) \longrightarrow H^1(G, \mathbb{Z}_2) \xrightarrow{Sq^2} H$$

Gu–Wen

$$H^4_{GW}(G) \longrightarrow H^2(G, \mathbb{Z}_2) \xrightarrow{\beta \circ Sq^2} H^5$$

Consider $\mathrm{Fer}(n) :=$ n real Majorana chiral fermions.
Then $\mathrm{Aut}(\mathrm{Fer}(n)) = O(n)$.

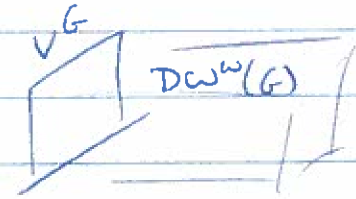
V_g is ordinary $\longleftrightarrow$ $g \in SO(n)$ (w_1)

can choose $V_g \otimes V_h \underset{\text{even}}{\cong} V_{gh}$ $\longleftrightarrow$ lift to $\mathrm{Spin}(n)$ w_2

associator α $\longleftrightarrow$ $\frac{p_1}{2}$.

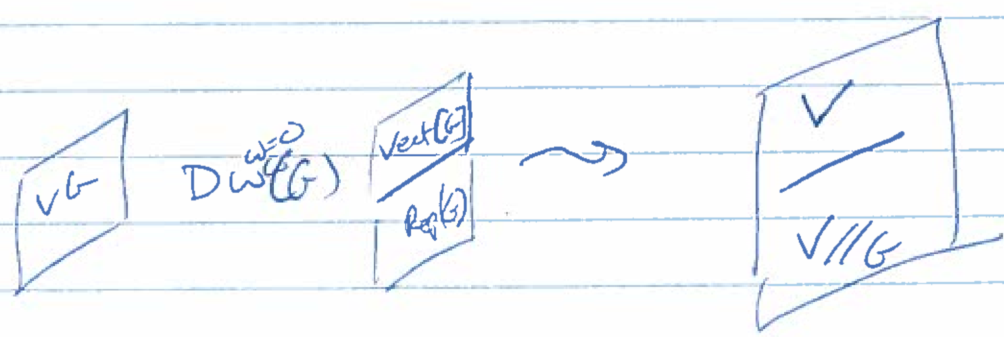
So $\frac{p_1}{2}(24)$ is the anomaly for $Co_0 \circlearrowright \mathrm{Fer}(24)$.

Suppose $G \subset V$ and you orbifold $\omega|_G$.
Then you get a new CFT called the orbifold
$V/\!/G$. It has various descriptions depending on
your preferred model. Here is a 3D one:
The fixed subalgebra V^G defines a chiral
conformal boundary condition of 3D G-Dijkgraaf–Witten
theory w/ cocycle $\omega|_G$:



$DW^\omega(G)$ is of Turaev-Viro type for fusion
category $Vect^\omega[G] = \{G\text{-graded vector spaces}, \omega = \text{associator}\}$,
meaning it admits a topological B.C. ~~that~~
~~[illegible]~~ described by that fusion cat. Iff $\omega = 0$,
it admits another top. b.c. like $Rep(G)$. These
b.c.s are Morita-equivalent via $Rep(G) \subset Vect \otimes Vect$

Consider the sandwich:



A special case is when $G = A$ is abelian. Then $\mathrm{Rep}(A) = \mathrm{Vec}[A^\vee]$ where $A^\vee$ is the Pontryagin dual group. Then you "trade" A for $A^\vee$.

Example:

$\to$ $\{\pm 1\} \circlearrowright \mathrm{Fer}(24)$, anomaly trivializes.

Duncan showed that $\mathrm{Fer}(24)/\!/\{\pm 1\}$ admits $\mathcal{N}=1$ SUSY, and that each choice of SUSY breaks $\mathrm{Aut}(\mathrm{Fer}(24)/\!/_{\pm 1}) = SO^+(\mathbb{R}^{24})$ to Co_0.

Most famous example:

Each even lattice determines a lattice CFT V_L w/
$$\mathrm{Aut}(V_L) \supseteq (L \otimes U(1)) . \mathrm{Aut}(L).$$
Let $L = $ Leech. Then ~~the~~ central $-1 \in Co_0$ lifts to $\mathrm{Aut}(V_L)$, where it acts nonanomalously.

$V^\natural := V_{\text{Leech}} /\!/ \mathbb{Z}_2$ is the famous moonshine module that won Richard a fields ~~medal~~ medal.

Some calculations + Noether Thm $\Rightarrow$ $\mathrm{Aut}(V^\natural)$ is finite. In fact:
$$\mathrm{Aut}(V^\natural) = \text{Monster. of order}$$
$$10^{54} \sim \quad 2^{46} . 3^{20} . 5^9 . 7^6 . 11^2 . 13^3 . 17 . 19 . 23 . 29 . 31 . 41 . 47 . 59 . 71 .$$

The <u>moonshine anomaly</u> is the ~~[illegible]~~ anomaly of Monster $\curvearrowright V^\natural$.

<u>Thm (in progress)</u> The moonshine anomaly is not zero. In particular, $H^4(\text{Monster}, \mathbb{Z}) \neq 0$. It has order ~~[illegible]~~ $2^k \cdot 3$ where
$$1 \leq k \leq 6$$
Almost surely $k=3$, but I need to do more calculations.

Let me explain how I know this. The large primes are easy to handle — I want to explain where the maximal subs containing the p-Sylows for small p come from.

Suppose A is abelian and normal in E, with $E/A = G$, i.e. $E = A.G$. To classify the extension you need:

- an action of G on A
- a cocycle $\alpha \in H^2(G, A)$.

Suppose $E \curvearrowright V$ s.t. $\omega|_A = 0$.
What acts on $V/\!/_A$? [c.f. Bhardwaj–Tachikawa]
Easiest case: A is cyclic. Then, since $\omega|_A = 0$, ω defines a class
$$\beta \in H^2(G, H^2(A, \mathbb{Z})) = H^2(G, A^\vee).$$

So you get a new extension

$$E^\vee = A^\vee . G$$

classified by β. There is a new anomaly.

~~See the~~

Think of this as "finite group T-duality":

$$\begin{array}{ccc} pt/A & \longrightarrow & pt/E \\ & & \downarrow \\ & & pt/G \end{array}$$

total space of a "torus" — really pt/A — bundle.

ω is a "B-field" on pt/E.

In cohomological algebra: (α, ω) is classified by:

$$\alpha \in C^2(G, A), \quad \beta \in C^2(G, A^\vee)$$
$$\text{s.t. } d\alpha = d\beta = 0, \quad \text{and}$$

$$\eta \in C^4(G, \mathbb{Z}) \quad \text{s.t.} \quad d\eta = \square(\underbrace{\alpha \cup \beta}_{\in C^4(G,\, A \otimes A^\vee) \to C^4(G,\, \mathbb{Z}/p)})$$

Bockstein

modulo

$$\alpha \sim \alpha + d\alpha', \quad \beta \sim \beta + d\beta'$$
$$\eta \sim \eta + d\eta' + \square(\langle \alpha \cup \beta' \rangle + \langle \alpha' \cup \beta \rangle + d\langle \alpha' \cup \beta' \rangle)$$

+ corrections coming from nonlinearity of $\square$

T-duality = switch $\alpha \Leftrightarrow \beta$.

Look at $\mathrm{Aut}(V_{\mathrm{Leech}}) = (\mathrm{Leech} \otimes U(1)).Co_0$.

The conj. classes $\subset Co_0$ of prime order that act w/o fixed points are:

$$\underset{\text{central}}{-1}, \quad 3A, \quad 5A, \quad 7A, \quad 13A$$

and lift to $\mathrm{Aut}(V_{\mathrm{Leech}})$ w/ non-anomalous action and with normalizers $\cong$

$$2^{24}.\underbrace{Co_0}_{2.Co_1 \ni g}, \quad 3^{12}.(\underset{g}{6}.Suz.2),$$

$$2.(A_7 \times 7{:}6).2$$

$$5^6{:}(4 \times 2J_2), \quad 7^4{:}(3 \times 2S_7), \quad 13^2{:}(3 \times 4S_4)$$

For any of these elements,

$$V_{\mathrm{Leech}} /\!/ \mathbb{Z}_p \cong V^\natural.$$

The T-dual groups to these normalizers are maximal subgps of $V^\natural$ of shape

$$2^{1+24}.Co_1, \quad 3^{1+12}.2Suz2, \quad 5^{1+6}{:}4 \times 2J_2$$

$$7^{1+4}{:}(3 \times 2S_7), \quad 13^{1+2}{:}(3 \times 4S_4).$$

These contain the p-Sylows for the small p.

So, to calculate ~~of~~ the moonshine anomaly, it is enough to restrict it to those subgps, but then you get something T-dual to the restriction of the anomaly for V_{Leech}.
~~I've calculated~~ ~~the V_{Leech} value for~~
Using this, I've calculated for $p \neq 2$, and achieved the (admittedly mild) bounds for $p = 2$.

Thank you.