Higher categories, generalized cohomology, and Condensed matter

Talk at UC Berkeley, 15 Nov 2017, by Theo Johnson-Freyd

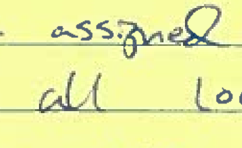
My goal for this talk is to tell you about an ongoing project joint w/ Davide Gaiotto and, time permitting, a related project joint w/ Mike Hopkins.

(0) The overall question is:

what is the classification of (topological) states of matter?

I hope to convince you that this is a fascinating question in pure algebra, with connections to Brauer groups, fusion categories, vertex algebras, ...

So what are the basic objects? The picture physicists have of a condensed matter system is as follows. I have some d-dimensional lattice — d is the space dimension, so that spacetime is $(d+1)$-dim — and a "local hilbert space" attached to each site in the lattice. The hilbert space for the quantum system assigned to some region is the tensor product of all local Hilbert spaces for all sites in the region.



You should think of the ~~[illegible]~~ sites as the nuclei of the atoms in a slab of metal, and the local hilbert spaces as the possible states of the electrons bound to that nucleus. In examples, it often makes sense to also have hilbert spaces at edges or faces in the lattice. But I can always coarse-grain my system and assign all degrees of freedom in each fundamental domain to one vertex.

To specify the dynamics of the system, I should choose a Hamiltonian. My rule will be that the Hamiltonian should be quasi-local: it should be a translation-invariant sum of terms each of which only includes interactions of sites that are w/in some distance of each other. Typically this interaction length scale is ~ the microscopic lattice spacing.

The first basic rule of condensed matter is to ask only *long-distance*, *low-energy* questions. This means, for example, that I may always zoom out, coarse-graining the lattice to some mesoscopic scale. Subject to some regularity conditions on the spectrum of the Hamiltonian, it means that the physics I care about is entirely determined by the *bottom* of the spectrum.

A *gapped* system is one where the bottom of the spectrum is as simple as possible: for any (macroscopic) region, the bottom eigenspace — the ground states — should be finite-dimensional, and there should be some positive gap before the ~~[illegible]~~ next eigenvalue, and these shouldn't change under mild changes to the region (say, enlarging it in a homotopically trivial way).

Actually, there's a lot of subtleties to this, because "the spectrum" depends heavily on the choice of boundary condition. I will ignore this subtlety.

A <u>phase</u> of gapped matter is an equivalence class under changes that do not "close the gap". Gapped phases are the things I and many others want to classify.

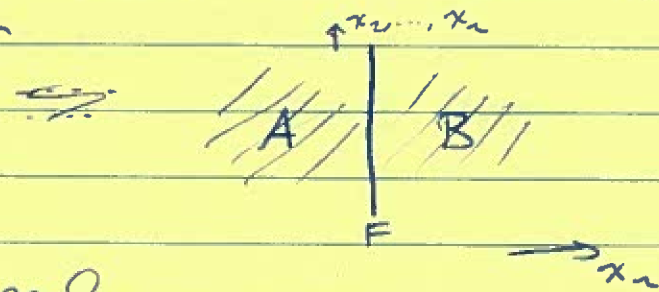
(1) There are some basic operations you can do to gapped phases. The set of gapped systems has a commutative and associative operation of <u>stacking</u>: just tensor the two systems together, with no interactions. This is commutative and associative if you work w/ microscopic descriptions of systems. But it will surprise no mathematician that if you work up

to phase, there can be homotopical information. For example, an *invertible phase* is not strictly invertible under stacking: it is a phase A s.t. $\exists B$ s.t. $A \otimes B$ is in the trivial *phase*. The choice of trivialization can include data.

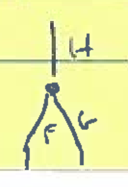
Example: The space of *$(-1+1)$-dim* invertible phases is homotopic to a circle $\mathbb{C}^\times$. A $(0+1)$ dim invertible phase is a complex line, so if you are bosonic, $\pi_0((0+1)\text{ dim phases}) = 0$.

I will write ~~[illegible]~~ $GP^\times_n$ for the space of n-dim invertible gapped phases
$\nwarrow$ spacetime: $n = d+1$.

Another ingredient that plays an important role is the notion of *defect*. This should be a system, translation-inv. in the $x_2, \ldots, x_n$ directions, living on



again gapped.
I also assume it is mildly topological: you should be able to bend the defect a little bit, as then you can draw systems like



, ...

Let's take $A = B$ invertible. Then I claim the "topological"ness of gapped defects is automatic:

<u>Proposition</u>: (1) {Gapped defects in an invertible n-dim ~~system~~ phase}

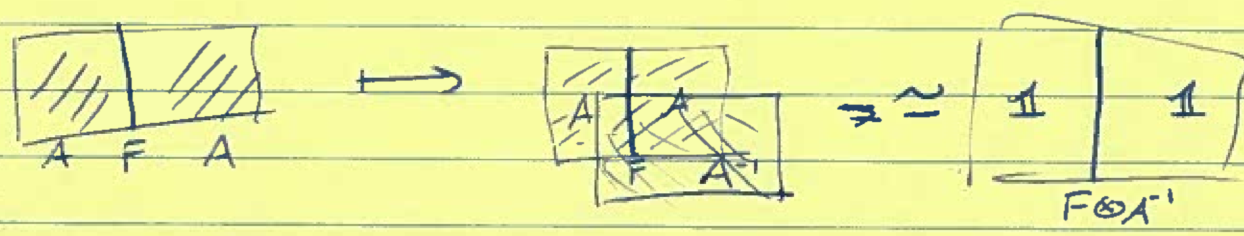
$\cong$

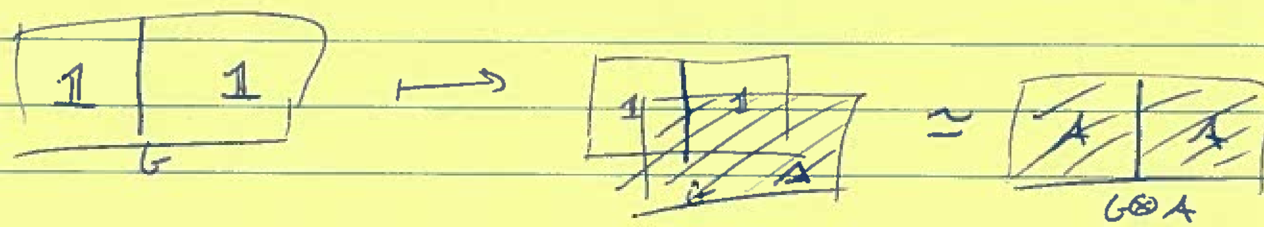
(2) {Gapped defects in the trivial n-dim phase}

$\cong$

(3) {$(n-1)$-dim gapped systems}.

<u>Pf</u>: $2 \Leftrightarrow 3$ is obvious. For $(1) \Leftrightarrow (2)$,





~~topol~~

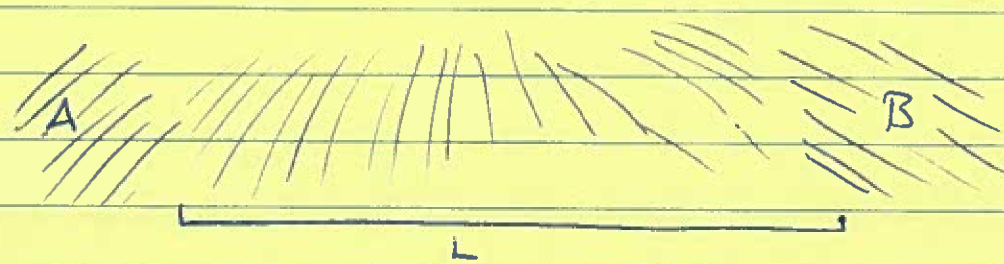
Defects can be composed. (If A is not invertible, then composition of defects in A is homotopy-assoc but not commutative. If A is invertible, then you get stacking of $(n-1)$-dim systems, so it is commutative.) In particular, you can talk about <u>invertible</u> defects.

<u>Proposition</u>: {Invertible defects separating A and B}

IS

{paths from A to B in the space of gapped systems}.

<u>Pf</u>: Suppose given a path ~~...~~ of systems from A to B. Build a system which looks like A at $x_1 \ll 0$, like B at $x_1 \gg 0$, and runs <u>very slowly</u> along the path as x_1 changes sign:

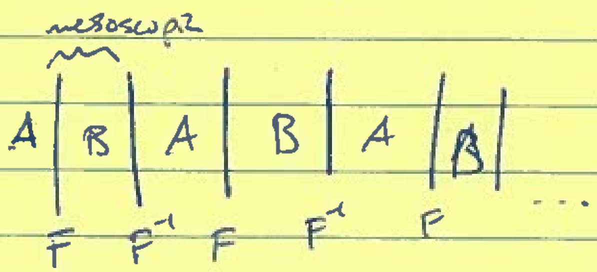


$L \gg$ microscopic length scale.

Now zoom out to get a defect

i.e. make L "mesoscopic"

Conversely, given an invertible defect F consider a system of slabs of A and B separated by F and F^{-1}:



$\cong$ A Bring $FF^{-1} \simeq \mathbb{1}$

$\cong$ B Bring $F^{-1}F \simeq \mathbb{1}$.

An immediate corollary:

<u>Thm [DG + Kitaev, TJF]</u> The spaces $GP^\times_n$ form an Ω-spectrum.

Let G be a group. A <u>G-protected phase</u> is a gapped phase w/ a G symmetry which is trivial if you ignore the G symmetry. To a mathematician, the following is a tautology:

$$\pi_0 \{ G\text{-protected } n\text{-dim systems} \} = \tilde{H}^n(BG;\ GP^\times_\bullet)$$

$\uparrow$ reduced coh.

But Davide and I give a constructive description of this equivalence in terms of mesoscopic systems w/ a σ-model w/ target G. Indeed, by Kock–Kristensen–Madsen, $H^n(BG;\ GP^\times)$ has a cocycle model w/ cochains

$$\prod_k C^{n-k}_{gp}(G;\ \pi_0 GP^\times_k)$$

and an "upper triangular" differential, and such cocycles directly build mesoscopic systems realizing the G-protected phase.

Moreover, ~~[illegible]~~ spectra are determined by their homotopy groups together w/ some connecting maps, which are necessarily stable cohomology operations. The condensed matter theorists have calculated enough to determine the low-dimensional ~~[illegible]~~ piece of GP:

Thm: For bosonic resp fermionic matter,

$$GP^{\times}\langle -3\rangle = \Sigma I_{\mathbb{Z}} MSO\langle -3\rangle \text{ resp } \Sigma I_{\mathbb{Z}} MSpin\langle -3\rangle$$

Presumably you can drop the "$\langle -3\rangle$"s. This confirms a prediction of Kapustin–Thorngren.

Example: The generator of

$$\cancel{\pi_{-3}}\quad \pi_{-3}\, \Sigma I_{\mathbb{Z}} MSO = \mathbb{Z}$$

is the E_8 phase. Note that E_8 does not define an (oriented) 3d TFT:

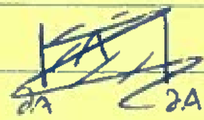
$\boxed{\text{top. phases of matter and TFTs are different}}$.

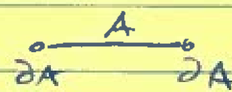
The E_8 phase is so-named because it admits massless (chiral) edge modes ~~[illegible]~~ described by the $E_{8,1}$ VOA.

(2) ~~[illegible]~~ How about non-invertible gapped phases?

(0+1)d: Vect (or SVect if you allow fermions).

(1+1)d: As far as anyone can tell, every (1+1)d gapped phase admits a gapped boundary condition. Let's choose a b.c. ∂A for the (1+1)d phase A. Consider a slab of A w/ b.c. ∂A:



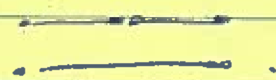


This ~~has~~ → is some f.d. vector space of ground states. Let's call that space $\underline{A}$. (Of course, it depends on the b.c.)

Consider putting two slabs together.

$\underline{A} \otimes \underline{A}$.

Compare w/ wider slab:



You can move between these systems adiabatically, either breaking or reforming the bond between them. This defines maps

$$Y \quad \begin{matrix} \underline{A} \otimes \underline{A} \\ \downarrow \\ \underline{A} \end{matrix} \qquad \begin{matrix} \underline{A} \\ \downarrow \\ \underline{A} \otimes \underline{A} \end{matrix} \ . \quad Y$$

These satisfy $\phi = |$ and $\underbrace{X = H = N}$.

Defn: A condensable algebra in a 1-cat D is an object w/

Exercise: condensable ⇒ associative.

N.B.: No unit assumed.

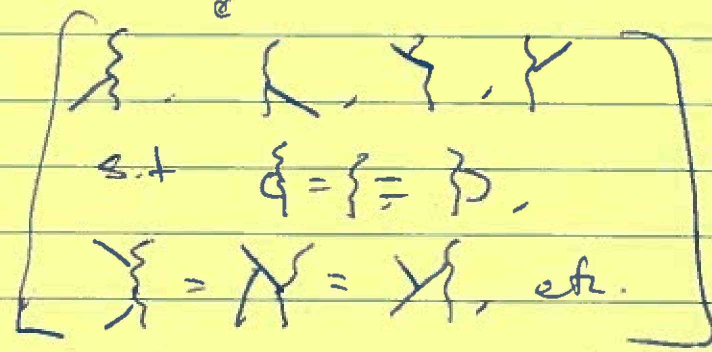
Thm [DG + TJF]

(a) Suppose $\mathcal{V}$ admits splittings of idempotents. Then there is a sym $\otimes$ bicat

$$\mathrm{CondAlg}(\mathcal{V})$$

whose objects are condensable algebras and morphisms are condensable bimodules.

(b) When $\mathcal{V} = \mathrm{Vec}^{f.d}$,
$\mathrm{CondAlg}(\mathrm{Vec}^{f.d})$ is bicategory of 2-dualizable objects in Alg

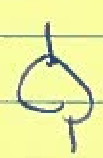


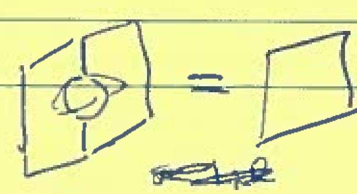
Alg = bicat of assoc. unital algs + (unital) bimods.

But the earlier analysis shows:

$$\left\{ \begin{array}{c} \text{Gapped (1+1)d phases} \\ \text{admitting gapped } \partial \end{array} \right\} = \mathrm{CondAlg}(\mathrm{Vec}^{f.d}).$$

Defn.
If $\mathcal{V}$ is a weak n-cat, relax condensability as follows: (*) Frobenius law holds up to higher "Frobenators", (**) replace

 $= 1$ w/

 s.t. 

or higher-dim version thereof.

[This defines a <u>condensation</u>. In a weak n-cat, a <u>condensation</u> $X \twoheadrightarrow Y$ is

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

together w/ a condensation of $gf \circlearrowright$ onto id_Y.]

<u>Thm in progress</u>:

(a) If V is a sym ⊗ weak n-cat such that [condensation version of "all idempotents split"], then $\exists$ sym ⊗ weak $(n+1)$-cat $\mathrm{CondAlg}(V)$ such that [cond. vers. of ...].

(b) If additionally V admits all duals, so does $\mathrm{CondAlg}(V)$.

This gives a higher cat of phases-that-admit-gapped-boundary, or, equivalently, "phases that can be condensed from the vacuum".

By (2), and the cobordism hypothesis, we have

$$\left\{\begin{matrix}\text{top. phases of matter condensable} \\ \text{from the vacuum}\end{matrix}\right\} = \left\{\begin{matrix}\text{TFTs condensable} \\ \text{from the vacuum}\end{matrix}\right\}.$$

(3) The zeroth example of a phase that does not admit a gapped b.c. is a fermionic vector space. The E_8 phase is another example.

Deligne's "existence of fibre functors" says that SVec, not Vec, is distinguished as an "algebraically closed" category, like $\mathbb{R} \rightsquigarrow \mathbb{C} = \overline{\mathbb{R}}$.

<u>Conjecture:</u> The "ultimate" classification of $(d+1)$-dim gapped phases will be by an $(d+1)$-categorical "algebraic closure" of $\mathrm{CondAlg}^{d}_{}(\mathrm{Vect})$.

If so, then the "ultimate" spectrum $GP^{\times}_{\bullet}$ will be not $\Sigma I_{\mathbb{Z}} MSpin$ but $\Sigma I_{\mathbb{Z}} MFr$ = "Anderson dual to spheres".

<u>Conjecture:</u> When $d=1$, $\mathrm{CondAlg}(\mathrm{SVec})$ <u>is</u> algebraically closed.

Proving this is work in progress w/ M. Hopkins.