MATH 3032: Abstract Algebra

Assignment 5

due 1 April 2025, end of day

Homework should be submitted as a single PDF attachment to `denisalja@dal.ca`. Please title the file in a useful way, for example `Math3032_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. The purpose of homework assignments is to learn.

In all questions below, F denotes some field.

1. Let $I \subset F[x, y]$ denote the ideal generated by $\{-x^3 + y, x^2y - y^2\}$.
   (a) Show that $\{x^3 - y, x^2y - y^2, xy^2 - y^2, y^3 - y^2\}$ is a reduced Gröbner basis for I with respect to the ordering $x \gg y$.
   (b) Determine whether $x^6 - x^5y \in I$.
2. Consider the ideal $I = (x^2 - y, x^2y - z) \subset F[x, y, z]$.
   (a) Show that $\{x^2 - y, y^2 - z\}$ is a reduced Gröbner basis for I with respect to the ordering $x \gg y \gg z$.
   (b) Show that $\{x^2 - y, z - y^2\}$ is a reduced Gröbner basis for I with respect to the ordering $z \gg x \gg y$.
   (c) Show that $\{y - x^2, z - x^4\}$ is a reduced Gröbner basis for I with respect to the ordering $y \gg z \gg x$.
3. Run Buchberger's Algorithm to find a reduced Gröbner basis for the ideal $(x-y^3, -x^2+xy^2) \subset F[x, y]$ with respect to the ordering $x \gg y$.
4. Run Buchberger's Algorithm to find a reduced Gröbner basis for the ideal $(x^2y-y^2, x^3-xy) \subset F[x, y]$ with respect to the ordering $x \gg y$.
5. Run Buchberger's Algorithm to find a reduced Gröbner basis for the ideal $(x^2+xy+z, xyz+z) \subset F[x, y, z]$ with respect to the ordering $x \gg y \gg z$.
6. The following system of equations has two solutions over $\mathbb{C}$:

$$x^2 = yz + 3, \qquad y^2 = xz + 4, \qquad z^2 = xy + 5.$$

   Find them.