MATH 2025: Introductory Analysis

Assignment 2

due February 6

Homework should be submitted *on paper* either in class or to the box by the instructor's office. Proofs do not need to be — and usually should not be — formatted like in a high school logic or geometry class. Rather, they should be formatted as clear mathematical writing: short, complete sentences assembled into paragraphs. Don't forget to "sign post": be clear about what you are proving, what still needs to be proved, what has been proved, and the like.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. The purpose of homework assignments is to learn.

Exercises are primarily drawn from *Introduction to Real Analysis* by R.G. Bartle and D.R. Sherbert.

1. Write the following sets as unions of intervals. You should show your work but you do not need to write careful proofs:
   (a) $\{x \in \mathbb{R} \text{ s.t. } x^3 > 3x + 4\}$
   (b) $\{x \in \mathbb{R} \text{ s.t. } 1 \le x^2 \le 4\}$
   (c) $\{x \in \mathbb{R} \text{ s.t. } 1/x \le x\}$
   (d) $\{x \in \mathbb{R} \text{ s.t. } 1/x < x^2\}$

2. (a) Suppose that $c > 1$ and that $m, n \in \mathbb{N}$. Show that $c^m > c^n$ iff $m > n$.
   (b) Suppose that $0 < c < 1$ and that $m, n \in \mathbb{N}$. Show that $c^m > c^n$ iff $m < n$.

3. Show that $\max(x, y) = \frac{1}{2}\big(x + y + |x - y|\big)$ for all $x, y \in \mathbb{R}$.

4. Find the inf and sup of the following sets, or show that they do not exist:
   (a) $\{x \in \mathbb{R} \text{ s.t. } 2x + 5 > 0\}$
   (b) $\{x \in \mathbb{R} \text{ s.t. } x^2 - 2x - 5 < 0\}$
   (c) $\left\{1 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

5. (a) Show that if $A, B \subset \mathbb{R}$ are bounded sets, then $A \cup B$ is bounded, and find a formula for $\sup(A \cup B)$ in terms of $\sup(A)$ and $\sup(B)$.
   (b) Use induction to show that if $A_1, \ldots, A_k \subset \mathbb{R}$ is a finite collection of bounded sets, then their union $\bigcup_{i=1}^{k} A_k$ is bounded.
   (c) Give an example of an infinite collection of bounded sets whose union is unbounded.

6. Suppose that $S \subset T \subset \mathbb{R}$ and that T is bounded and that S is nonempty.

(a) Show that S is bounded.

(b) Show that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

7. Let $S \subset \mathbb{R}$ be nonempty and bounded. Given $a \in \mathbb{R}$, define $a + S := \{a + s : s \in S\}$ and $aS := \{as : s \in S\}$.

   (a) Show that $a+S$ is bounded and that $\inf(a+S) = a+\inf(S)$ and $\sup(a+S) = a+\sup(S)$.

   (b) Show that aS is bounded. Show that if $a \geq 0$, then $\inf(aS) = a \inf S$ and $\sup(aS) = a \sup S$. What if $a < 0$?

8. Prove that if $y \in \mathbb{R}$, then there is a unique $n \in \mathbb{Z}$ such that $n \leq y < n + 1$.

9. Compute the following sets:

   (a) $\bigcap_{n=1}^{\infty} [0, \frac{1}{n}]$

   (b) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

   (c) $\bigcap_{n=1}^{\infty} (n, \infty)$

   (d) $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}]$

   (e) $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$

   (f) $\bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$

   (g) $\bigcap_{n=1}^{\infty} (0, 1 + \frac{1}{n})$