MATH 2025: Introductory Analysis

Assignment 3

due March 9

Homework should be submitted *on paper* either in class or to the box by the instructor's office. Proofs do not need to be — and usually should not be — formatted like in a high school logic or geometry class. Rather, they should be formatted as clear mathematical writing: short, complete sentences assembled into paragraphs. Don't forget to "sign post": be clear about what you are proving, what still needs to be proved, what has been proved, and the like.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. The purpose of homework assignments is to learn.

Exercises are primarily drawn from *Introduction to Real Analysis* by R.G. Bartle and D.R. Sherbert.

1. List the first five terms of the following inductively defined sequences. You do not need to justify your answers.
   (a) $x_0 := 1$ and $x_n := 3x_{n-1} + 1$
   (b) $y_0 := 2$ and $x_{n+1} := \frac{1}{2}(y_n + \frac{2}{y_n})$.
   (c) $z_0 := 1$, $z_1 := 2$, and $z_n := (z_{n-1} + z_{n-2})/(z_{n-1} - z_{n-2})$.
2. Use the definition of the limit of a sequence to establish the following:
   (a) $\lim_{n\to\infty} \dfrac{2n}{n+2} = 2.$
   (b) $\lim_{n\to\infty} \dfrac{\sqrt{n}}{n+1} = 0.$
   (c) $\lim_{n\to\infty} \dfrac{(-1)^n n}{n^2+1} = 0.$
3. Show that $\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right) = 0.$
4. Give an example of two divergent sequences whose sum converges. Give an example of two divergent sequences whose product converges.
5. Show that the following sequences are not convergent:
   (a) 2^n
   (b) $(-1)^n n^2$
   (c) $\dfrac{(-1)^n n}{n+1}$

(d) $1 + (-1)^n + \dfrac{1}{n}$

(e) $\sin\left(\dfrac{n\pi}{4}\right)$

6. Suppose that x_n is a convergent sequence with $\lim x_n = x \neq 0$. Prove that there are at most finitely many values of n such that $x_n = 0$. Define the sequence y_n by:

$$y_n = \begin{cases} 0, & x_n = 0, \\ \frac{1}{x_n}, & x_n \neq 0. \end{cases}$$

Prove that $\lim y_n = \frac{1}{x}$.

7. Use the Squeeze Theorem to determine the limits of the following sequences:

(a) n^{1/n^2}

(b) $(n!)^{1/n^2}$

8. Define $x_0 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$. Show that (x_n) is bounded and monotone (hint: use induction). Find the limit.

9. Consider the sequence

$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Is y_n convergent or divergent? Justify your answer.

10. Suppose that X is a sequence with the following property: for every subsequence of X, there is a subsubsequence that converges to 0. Prove that X converges to 0.

11. Prove that if (x_n) and (y_n) are bounded sequences, then

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

and find an example in which the inequality is strict (i.e. in which the two sides are not equal).