MATH 2025: Introductory Analysis

Assignment 4

due March 20

Homework should be submitted *on paper* either in class or to the box by the instructor's office. Proofs do not need to be — and usually should not be — formatted like in a high school logic or geometry class. Rather, they should be formatted as clear mathematical writing: short, complete sentences assembled into paragraphs. Don't forget to "sign post": be clear about what you are proving, what still needs to be proved, what has been proved, and the like.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. The purpose of homework assignments is to learn.

Exercises are primarily drawn from *Introduction to Real Analysis* by R.G. Bartle and D.R. Sherbert.

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = \ell$ if and only if $\lim_{x\to 0} f(x+c) = \ell$.

2. Let $A := (0, a)$ for some $a > 0$, and let $f : A \to \mathbb{R}$ be the function $f(x) = x^2$. Show that for any $x, c \in A$, we have $|f(x) - c^2| \le 2a|x - c|$. Use this inequality to prove that $\lim_{x\to c} f(x) = c^2$ for all $c \in A$.

3. Use the ϵ-δ definition of limit to show that $\lim_{x\to 1} \frac{x^2}{1+x^2} = \frac{1}{2}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{R} \end{cases}$$

   Show that f has a limit at $c = 0$ and compute it, and show that f does not have a limit at any nonzero c.

5. Prove that $\lim_{x\to 0} \cos(1/x)$ does not exist but that $\lim_{x\to 0} x\cos(1/x)$ does.

6. Let $f, g : A \to \mathbb{R}$ be functions, and suppose that c is a cluster point of A. Suppose that f is bounded on a neighbourhood of c and that $\lim_{x\to c} g(x) = 0$. Show that $\lim_{x\to c} f(x)g(x) = 0$.

7. A function $f : \mathbb{R} \to \mathbb{R}$ is called *additive* if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   (a) Suppose that f is additive and that $\lim_{x\to 0} f(x)$ exists. Prove that under this hypothesis, $\lim_{x\to 0} f(x) = 0$.

(b) Prove under the same hypothesis that $\lim_{x\to c} f(x) = f(c)$.

(c) Suppose that f is additive and that there exists a c such that $\lim_{x\to c} f(x)$ exists. Prove that $\lim_{x\to 0} f(x)$ exists.

(d) ! Prove that there exists an additive function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 0} f(x)$ does not exist. Hint: $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

8. Prove that if $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|$.

9. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\nearrow 0} f(x)$ exists but $\lim_{x\searrow 0} f(x)$ does not exist.