MATH 2025: Introductory Analysis

Assignment 5

due April 9

Homework should be submitted *on paper* either in class or to the box by the instructor's office. Proofs do not need to be — and usually should not be — formatted like in a high school logic or geometry class. Rather, they should be formatted as clear mathematical writing: short, complete sentences assembled into paragraphs. Don't forget to "sign post": be clear about what you are proving, what still needs to be proved, what has been proved, and the like.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. The purpose of homework assignments is to learn.

Exercises are primarily drawn from *Introduction to Real Analysis* by R.G. Bartle and D.R. Sherbert.

1. Show that the sum of two monotonic-increasing functions is monotonic-increasing. Give an example of two monotonic-increasing functions whose product is not monotonic-increasing.

2. Show that $f(x) := 1/x$ is uniformly continuous on (a, ∞) for any $a > 0$, but that it is not uniformly continuous on $(0, \infty)$.

3. Show that a uniformly continuous function on a bounded set is necessarily bounded. Give an example of a function on a bounded set which is (non-uniformly) continuous but unbounded.

4. Compute the following pointwise limits. Is the convergence uniform on $[0, \infty)$? Is the convergence uniform on $[1, \infty)$?

   (a) $\lim_{n\to\infty} \frac{nx}{1+nx}$

   (b) $\lim_{n\to\infty} \frac{\sin nx}{1+nx}$

5. Show that the sequence $\frac{x^n}{1+x^n}$ does not converge uniformly on $[0, 2]$ by showing that its limit is not continuous on $[0, 2]$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Set $f_n(x) := f(x + \frac{1}{n})$. Show that f_n converges uniformly to f.

7. Can a sequence of unbounded functions converge pointwise? Can a sequence of unbounded functions converge uniformly? Can a sequence of unbounded functions converge pointwise to a bounded function? Can a sequence of unbounded functions converge uniformly to a bounded function? Can a sequence of bounded functions converge pointwise to an unbounded function? Can a sequence of bounded functions converge uniformly to an unbounded function?

8. ! Let $\mathbb{R}^n$ denote the space of n-dimensional vectors $\vec{v} = (v_1, v_2, \ldots, v_n)$. For each positive integer p, consider the function $\| - \|_p$ defined by:

$$\|\vec{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

   (a) Prove that for each $\vec{v}$, we have

$$\lim_{p \to \infty} \|\vec{v}\|_p = \sup_i |v_i|.$$

   In other words, writing $\|\vec{v}\|_\infty := \max\{|v_1|, |v_2|, \ldots, |v_n|\}$, prove that

$$\| - \|_p \to \| - \|_\infty$$

   converges pointwise.

   (b) Is the convergence uniform? What if you restrict the domain of the functions to just those $\vec{v}$ for which all coefficients are in $[-1, 1]$?