Lecture 10: Bounded Monotone Convergence

04. Feb. 2026

$\alpha := \sup(t^2 \leq 2)$ $\qquad \alpha^2 = 2$ $\qquad t^2 \leq 2 \quad \alpha \quad t^2 \geq 2$, β

$\beta := \inf(s^2 \geq 2)$

→ we want to show that $\alpha = \beta$: $\beta \to M \geq t$ s.t. $t^2 \geq 2$, $M \geq \alpha$ $\qquad s^2 \geq 2 \Rightarrow s \geq \alpha \Rightarrow \inf \geq \sup$ ($\beta \geq \alpha$)

$\alpha \to M' \leq s$ s.t. $s^2 \leq 2$, $M' \leq \beta$

wts: $\sup \geq \inf$, suppose not $\quad \sup < \inf \to \sup < c < \inf$ $\quad 2 \leq c^2$, $\Rightarrow c^2 \geq 2$ → contradicts the inf

"the n^{th} term thats bigger than the N^{th} term"

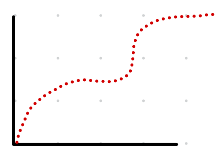
Warm up: $\lim x_n = x \Rightarrow |x_n| \to |x|$ $\qquad$ for all $\varepsilon > 0$, $\exists\, N$ s.t. $n \geq N$ $|x_n - x| < \varepsilon$

$\forall\, \varepsilon > 0$ eventually $n \geq N$ we will get $||x_n| - |x|| < \varepsilon$ → $= ||x_n| - |x||$

Bounded Monotone Convergence:

→ $|x_n| < M$ ← $-M < x_n < M$

for some M



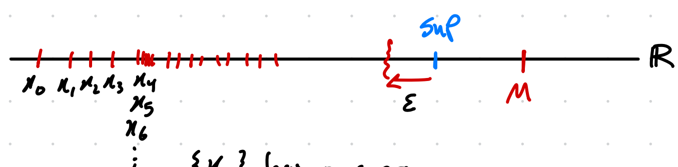
Definition: x_n is non decreasing if $x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$

Remark: " " increasing " $x_0 < x_1 < x_2 < \dots$

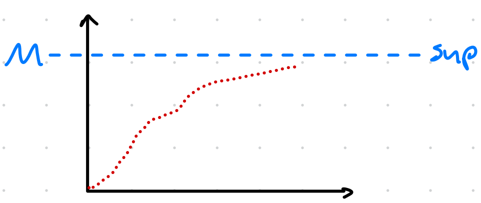
" " non-increasing " $x_0 \geq x_1 \geq x_2 \geq \dots$

→ Monotone = one of these

→ x_n is non-decreasing & bounded by M



$\{x_n\}$ has a supremum



Theorem: if x_n is monotone & bounded it converges to $\sup(x_n)$

Proof: Suppose $x_n \leq x_{n+1}$

wts: $\forall\, \varepsilon > 0$ eventually $n \geq N$, $|x_n - \sup| < \varepsilon$ → $-\varepsilon < x_n - \sup < \varepsilon$

$-\varepsilon + \sup < x_n < \varepsilon + \sup$

$\exists\, x_N$ s.t. $x_N \geq \sup - \varepsilon$

(otherwise $\sup - \varepsilon < x_n \leq \sup$ → $\sup - \varepsilon \leq x_N \leq x_{N+1} \leq x_{N+2} \leq \dots$

$n \geq N \Rightarrow \sup - \varepsilon < x_n - \varepsilon$ ■

Example: $x_n = \frac{1}{\sqrt{n}}$ → $x_n \geq 0$, → $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$ non increasing, yes it converges

Example: $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{2}{2^n}$ → increasing bounded by 2 it converges

Example: $h_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \longrightarrow$ increasing

$$\geq 1+\frac{1}{2}+\underbrace{\left(\frac{1}{4}+\frac{1}{4}\right)}_{\frac{1}{2}}+\underbrace{\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)}_{\frac{1}{2}}+\cdots$$

$\geq 1+\frac{n}{2} \longrightarrow$ not bounded, not convergent

Example: $\begin{cases} x_0 = 2 \\ x_{n+1} = 2+\frac{1}{x_n} \end{cases}$ $\quad x_n \geq 0$ (exercise induction)

$\Longrightarrow x_n$ is increasing, $x_0 \geq 2 \longrightarrow x_n \geq 2$

$\frac{1}{2} \geq \frac{1}{x_n} \longrightarrow 2+\frac{1}{x_n} \leq 2+\frac{1}{2} < \infty$

bounded

converges

$x_n \to x \quad x = ? \quad x_{n+1} = 2+\frac{1}{x_n} \longrightarrow x = 2+\frac{1}{x}$

$x_{n+1} \to x$

bc x_n, x_{n+1} converge

$x_n \neq 0$

$x^2 = 2x+1$

$x^2-2x-1=0$

$$\boxed{x = 1+\sqrt{2}}$$

Exercise 3.3.5

for $S_n \longrightarrow \sqrt{s}$; $s \in \mathbb{R}^+$

Example: $e_n = \left(1+\frac{1}{n}\right)^n = 1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\binom{n}{3}\frac{1}{n^3}+\cdots+\binom{n-1}{n-1}\frac{1}{n^{n-1}}+1$

$$= 1+1+\frac{1}{2!}\frac{n-1}{n}+\frac{1}{3!}\frac{(n-1)(n-2)}{n^2}+\cdots+\frac{(n-1)!}{(n-1)!\,n^{n-1}}$$

$$e_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = 1+1+\frac{1}{2!}\frac{n}{n+1}+\frac{1}{3}\frac{n(n-1)}{(n+1)^2}+\cdots+\frac{(n+1)!}{}$$

$$\boxed{e_{n+1} \geq e_n}$$

(I didn't get this part just look it up)

$e_n = 1+1+\cdots+\frac{1}{k!}\underbrace{(-)}_{\leq 1} \qquad n \geq 2$

$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!} \qquad n! \leq 2^n$

$\leq 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} \qquad e_n \longrightarrow e$

$\leq 1+2 = 3$