Lecture 11: Convergent Sequences

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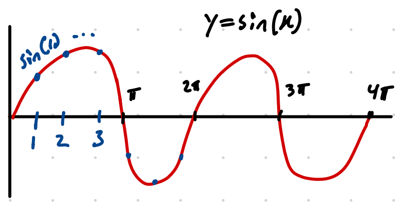
Recall: A sequence is a function $\mathbb{N} \to \mathbb{R}$
$n \mapsto x_n$

Observation about $\mathbb{N}$: $0<1<2<3\cdots$ pick any infinite subset $S \subseteq \mathbb{N}$; then there is a unique monotonic (increasing) bijection $\mathbb{N} \to S$

- $\to$ conversely any strictly monotonic function $\mathbb{N} \to \mathbb{N}$ has some image
- $\to$ Infinite subsets of $\mathbb{N} \longleftrightarrow$ strictly monotonic functions $\mathbb{N} \to \mathbb{N}$
- $\to$ Given $i \mapsto n_i$ a monotonic function $\mathbb{N} \to \mathbb{N}$ and a sequence $n \mapsto x_n$, get a subsequence $i \mapsto x_{n_i}$

Example: $x \mapsto \sin(n)$



- $\to$ example of a subsequence would be $n \mapsto \sin(2n)$, this is a subset of $n \mapsto \sin(n)$
- $\to$ another subsequence of $n \mapsto \sin(n)$ is the one that records only the positive values
  $\sin(0), \sin(1), \sin(2), \sin(7), \sin(8), \ldots$

- $\to$ non-eg: $n \mapsto \sin\left(\frac{n}{2}\right)$ is not a subset of $n \mapsto \sin(n)$
- $\to$ $\sin(0), \sin(2), \sin(1), \sin(4), \sin(3), \sin(6), \sin(5), \ldots$ not a subsequence because its out of order.

Theorem: Suppose $X: \mathbb{N} \to \mathbb{R}$ is a convergent sequence and Y is a subsequence. Then $\lim Y = \lim X$ (and in particular $\lim Y$ exists)

Proof: Write $x := \lim X$. The assumption that $X \to x$ means $\forall \varepsilon > 0, \exists K$ s.t. $\forall n > K, |x_n - x| < \varepsilon$
Suppose $Y_i = X_{n_i}$ (just to set notation)

- $\to$ Given ε, pick K as above (for x), set ℓ any number s.t. $n_\ell > K$
- $\to$ Then if $i > \ell$ then $n_i > n_\ell > K$ So $|Y_i - x| = |X_{n_i} - x| < \varepsilon$; why does such ℓ exist? b/c my subsequence is indexed by an infinite set and only finitely many n_i's will be less than K.

Corollary: Suppose X is a sequence s.t. exists subsequences Y, Z with $\lim Y \neq \lim Z$ then X diverges

Example: $x_n = (-1)^n + \frac{1}{2^n}$

$n=0 \quad x=2$
$n=1 \quad x=-\frac{1}{2}$
$n=2 \quad x=\frac{5}{4}$
$n=3 \quad x=-\frac{7}{8}$

$\to$ look at the subseq of even entries
$x_{2n} = 1 + \frac{1}{2^n} \longrightarrow 1+0=1$
$x_{2n+1} = -1 + \frac{1}{2^n} \longrightarrow -1+0=-1$

Converse of the definition of limit

TFAE:

- X does not converge to x $\quad x_n \not\to x$
- $\exists\, \varepsilon>0$ there are infinitely many n s.t. $|x_n - x| > \varepsilon$
- $\exists_{\varepsilon \text{ s.t. } \exists}$ subsequence of X, call it Y s.t. $|Y_i - x| > \varepsilon \quad \forall i$

Theorem: Every sequence admits a weakly monotonic subsequence.

Proof: Try to build an increasing subsequence in the most naive way:

- → Start w/ x_0, then pick the next entry above x_0, then the earliest entry above that and so on.
- → If this succeeds, great.
- → If this naive attempt fails, I must have hit a value of my sequence which is bigger than all its future.

Definition: A <u>peak</u> of X is an x_n s.t. $\forall\, m>n \;\; x_n > x_m$

- → if only finitely many peaks, just start a naive attempt after the last peak.
- → if there are infinitely many peaks, the the subsequence of peaks is monotonic-decreasing.

$\blacksquare$

Corollary: Every bounded sequence has a convergent seq.