Lecture 12: Convergent Sequences Cont.

11·Feb·2026

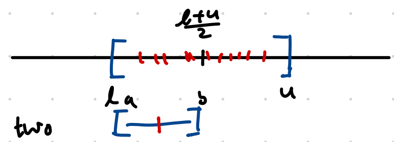
Bolzano–Weierstrass Theorem: every bounded sequence (in $\mathbb{R}$) contains a convergent subsequence.

Proof 1: we already proved that every sequence contains a monotonic subsequence. Any subsequence of a bounded sequence is bounded.

→ And so: any bounded sequence contains a bounded monotone subsequence.

→ we already proved that bounded monotone sequences converge. ■

Proof 2: Let (x_n) be a sequence bounded in $[\ell, u]$



→ $\because [\ell,u] = [\ell, \frac{\ell+u}{2}] \cup [\frac{\ell+u}{2}, u]$ at least one of these two halves contains a subseq of (x_n)

→ say $[a,b]$ does (either $a,b = \ell, \frac{\ell+u}{2}$ or $= \frac{\ell+u}{2}, u$

→ By exactly same argument, at least one of $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$ contains infinitely many entries in my sequence.

→ divide again, pick a half containing as many seq. points

⋮

→ we get a nested sequence of intervals

→ Build a new sequence of the left endpoints of these intervals (ℓ_n) $\ell_0 = \ell$
$\ell_1 =$ either ℓ or $\frac{\ell+u}{2}$

→ ℓ_n is monotonic increasing and bounded above by u.
(the collection of left endpoints is bounded so it has a supremum).

→ The collection of right endpoints is bounded so it has an infimum)

for each n, $\ell_n \le \sup\binom{\text{set of left}}{\text{end points}}$ $\inf(\text{set of } u_n\text{s}) \le u_n$ because every u_n is bigger than every ℓ_n

$|u_n - \ell_n| = \frac{u-\ell}{2^n}$ → so $0 \le \left|\inf\binom{\text{right}}{\text{endpoints}} - \sup\binom{\text{left}}{\text{endpoints}}\right| \le \frac{u-\ell}{2^n}$ for every n

→ let $x := \sup(\text{left endpoints}) = \inf(\text{right endpoints})$ **Goal:** build a subseq of (x_n) converging to this x → say (x_{n_i})

→ Know that the i^{th} interval $[\ell_i, u_i]$ contains ∞ entries from seq (x_n)
so I can definitely pick one "x_{n_i}" and pick it so that $n_i > n_{i-1}$.
I find a subseq $x_{n_i} \subseteq x_n$ s.t. $\ell_i \le x_{n_i} \le u_i$ → **claim:** this sequence converges to x

→ have seq (ℓ_i) of left endpoints
(u_i) of right endpoints $\quad x_n \supseteq (x_{n_i}) = (y_i)$

Know:
- $\ell_i \le y_i \le u_i$
- $\lim \ell_i = \lim u_i = x$

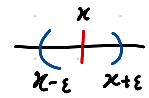
Squeeze Lemma: if Im given three sequences $(\ell_i), (y_i), (u_i)$ s.t. $\ell_i \le y_i \le u_i$ for every i and $\lim \ell_i = \lim u_i = x$ then $\lim y_i = x$

Proof: Given $\varepsilon > 0$ $\because \lim \ell_i = x$ I know $\exists N$ s.t. $|\ell_i - x| < \varepsilon$ $\forall i > N$ and $\because \lim u_i = x$ $\exists M$ s.t. $|u_i - x| < \varepsilon$ $\forall i > M$; let $i > \max(N, M)$

WTS: $|y_i - x| < \varepsilon$

- if $y_i \le x$ then $\ell_i \le y_i \le x$ but $|\ell_i - x| < \varepsilon$ so $|y_i - x| < \varepsilon$
- if $y_i \ge x$ then $u_i \ge y_i \ge x$ but $|u_i - x| < \varepsilon$ so $|y_i - x| < \varepsilon$

→ Given a sequence $(x_0, x_1, x_2, \dots)$

→ Saying "x is a <u>limit</u> of the above sequence" if $\forall \varepsilon > 0$ the sequence enters 

and never leaves; this also means there are only finitely many sequence entries <u>not</u> in $(x-\varepsilon, x+\varepsilon)$

→ x is an <u>accumulation point</u> of $(x_0, x_1, \dots)$ if $\forall \varepsilon > 0$ infinitely many sequence entries are in $(x-\varepsilon, x+\varepsilon)$
If x is a limit then it is an accumulation point, but not conversely. A seq has at most one limit but a seq can have many acc. pts.

↓

These ways of writing the definition are independent of how you order a sequence.

<u>Theorem:</u> x is an accumulation point of $(x_0, x_1, x_2, \dots) \Longleftrightarrow x$ is the limit of some subseq of (x_n).

<u>Theorem:</u> Any bounded sequence has an accumulation point.

<u>Example:</u> $\mathbb{Q}$ is countable in particular can find $\mathbb{N} \to \mathbb{Q}$ i.e. $\exists$ seq that hits every rational number

Every real number is an accumulation point of this seq.