Lecture 13: Finishing up Sequences

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Last time: A sequence can have many accumulation points i.e. subsequential limits
→ If X is a sequence, x is a
- acc. pt. if $\exists\, \varepsilon>0$, there are infinitely many n s.t. $X_n \in (x-\varepsilon, x+\varepsilon)$
- Subsequence limit if $\exists$ subsequence $Y \subseteq X$ s.t. $\lim Y = x$

Theorem: These are the same

Proof (strat): For each n, X enters $(x-\frac{1}{n}, x+\frac{1}{n})$ infinitely often so build $Y \subseteq X$ s.t. $Y_n \in (x-\frac{1}{n}, x+\frac{1}{n})$

Example: Since $\mathbb{Q}$ is countable there exists a sequence that hits every rational #.
if $r \in \mathbb{R}$ is any real number then $\forall \varepsilon$, $(r-\varepsilon, r+\varepsilon) \cap \mathbb{Q}$ is infinite → so r is an acc. pt. of X
r was arbitrary

Lemma: if $a<b$ in $\mathbb{R}$, then $(a,b)\cap\mathbb{Q}$ is infinite
→ Define $b-a =: w > 0$ $\frac{1}{w}>0$ pick $N \in \mathbb{N}$ s.t. $N > \frac{1}{w}$
→ So (Na, Nb) has width >1 so contains an integer M
→ So $\frac{M}{N} \in (a,b)$ Now subdivide and repeat

Let X be a bounded set. $\sup(X)$ might or might not be an acc. pt.

Example:

$X = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ → then $\sup(X) = 1 = \lim(X)$

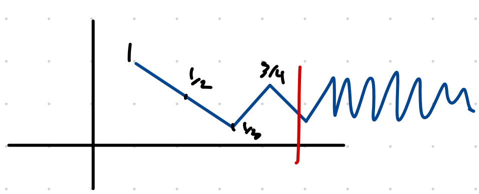
$X = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ → then $\sup(X) = 1$ but only acc. pt. is $\lim X = 0$

Remark: Bolzano-Weierstrass says that a bounded seq has at least one acc. pt. but maybe many.

- For bounded sequences, TFAE:
  → Seq converges (limit exists
  → $\exists!$ acc. pt.

Suppose given a bounded sequence $X = (x_0, x_1, x_2, \dots)$; define a new sequence whose nth term is
$$S_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\} = \sup\{x_m\} \quad m \geq n$$

$\frac{(-1)^n+1}{2} - (-1)^n\frac{1}{n}$ → n even: $1-\frac{1}{n}$
n odd: $0+\frac{1}{n}$



we are throwing away part of the set and getting a new supremum

$S_{n+1} \leq S_n$

$$S_n \geq S_{n+1} \geq S_{n+2} \geq \cdots \geq \inf(X)$$

→ is a decreasing and bounded sequence and so converges
→ $\lim_{n\to\infty} \sup(x_n, x_{n+1}, x_{n+2}, \dots) = \lim\sup(X)$

Theorem: Let X be a bounded sequence
i) $\lim\sup(X)$ is an acc. pt. of X
ii) $\lim\sup(X) = \sup(\text{acc. pts. of } X)$

! This tells you that for any sequence X, the set of accum. pts. of X has the property that it contains its own sup.

Ex.: So $\nexists\, X$ s.t. $\{\text{accum. pts. of } X\}$ is $(0,1)$

Proof: i) let $\ell := \lim\sup(X)$; by definition, $\forall \varepsilon$, $\exists n$ s.t. $\ell \leq S_n < \ell + \varepsilon$ thus $\exists\, N > n$ s.t.
$S_n \geq x_N > S_n - \varepsilon$ → so $\ell + \varepsilon > x_N > \ell - \varepsilon$, so these x_N's are a subseq converging to ℓ.

ii) Suffice to show if y is any accumulation point then $y \leq \limsup(X)$
(Given i)) for this suffice to observe that if $Y \subseteq X$ then $\limsup Y \leq \limsup X$

if X converges then $\limsup = \lim$

Corollary: for a bounded seq X TFAE:

i) $\lim X$ exists

ii) $\limsup X = \liminf X = \lim X$ in which case

proof: if $\lim X$ exists then it is the only accum. pt. of X so (i) $\Rightarrow$ (ii)

→ for any bounded X, acc. pts $(X) \subseteq [\liminf X, \limsup X]$

→ So if $\liminf = \limsup = \ell$ then accum. pts. $\subseteq \{\ell\}$

→ So $\exists!$ accm. pt. So (ii) $\Rightarrow$ (i)

Definition: A sequence $X = (x_0, x_1, \ldots)$ is Cauchy if $\forall \varepsilon\ \exists N$ s.t. $\forall m,n \geq N\ |x_m - x_n| < \varepsilon$

Lemma: If $\lim X$ exists, then X is Cauchy.

proof: since $\lim X$ exists, $\forall \varepsilon, \exists N$ s.t. $\forall n \geq N\ |x_n - \lim X| < \frac{\varepsilon}{2}$ so Δ ineq $\Rightarrow |x_n - x_m| < \varepsilon$ if $n,m \geq N$

Theorem: if X is Cauchy then X converges.

Proof strat: show that Cauchy $\Rightarrow \limsup = \liminf$

idea: use cauchy condition to show that $\forall \varepsilon, |\limsup - \liminf| < \varepsilon$

Application: A seq x_0 is called contractive if $\exists\, 0 < C < 1$ s.t. $|x_{n+1} - x_n| \leq C\,|x_n - x_{n-1}|$

Lemma: contractive sequences converge

proof: suppose $m \geq n$ then from Δ inequality $|x_m - x_n| \subseteq \sum_{k=1}^{m-n} |x_{n+k} - x_{n+k-1}| \leq \sum_{k=1}^{m-n} C^k |x_{n+1} - x_n|$ ★

$$\leq \sum_{k=1}^{\infty} C^k |x_{n+1} - x_n| = \frac{1}{1-C} |x_{n+1} - x_n| \leq C^n |x_1 - x_0|$$

→ Given ε find N s.t

$$\frac{C^N}{1-C} |x_1 - x_0| < \varepsilon$$

can do this b/c since $C < 1$ $C^N \cdot \frac{|x_1 - x_0|}{1-C} \to 0$ as $N \to \infty$

For this N ★ shows that X is Cauchy hence convergent.