Lecture 14: Functions

25. Feb. 2026

$\rightarrow$ Suppose $A \subset \mathbb{R}$; a real number c is in the <u>closure</u> of A if every neighbourhood $c \in U$ intersects A (non trivial)

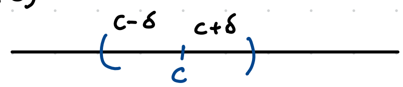
<u>Neighbourhood</u>: • a nbhd of c is an open interval containing c

• given $\delta > 0$, the δ-nbhd of c $V_\delta(c)$ is $(c-\delta, c+\delta)$

<u>equiv</u>: c is in the closure of A if

$\forall \delta > 0, \exists a \in A$ s.t. $|a - c| < \delta$

$\rightarrow cl(A) = \bar{A} =$ closure of A



<u>E.g.</u>: if $c \in A$ then $c \in \bar{A}$

<u>Definition</u>: c is a cluster point of A if $c \in \overline{A - \{c\}}$

$\rightarrow$ i.e. $\forall$ nbhd $c \in U$ $\exists a \in A$ w/ $a \neq c$ and $a \in U$

$\rightarrow$ i.e. $\forall \delta > 0$, $\exists a \in A$ s.t. $0 < |a - c| < \delta$

<u>E.g.</u>: $\text{cluster}(\{3,5,7\}) = \emptyset \rightarrow \text{Closure}(\{3,5,7\}) = \{3,5,7\}$

$\text{cluster}([3,7]) = [3,7] \rightarrow \text{Cl}([3,7]) = [3,7]$

$\text{Cluster}((3,7)) = [3,7] \rightarrow \text{Closure}((3,7)) = [3,7]$

<u>Goal</u>: Fix some domain $A \subset \mathbb{R}$

look at functions $f: A \to \mathbb{R}$

Care about the behaviour of f near but <u>not</u> at c $\rightarrow$ so might as well have $A \to A - \{c\}$

$\rightarrow$ Let $A \subset \mathbb{R}$ and $c \in \text{cluster}(A)$

and $f: A - \{c\} \to \mathbb{R}$

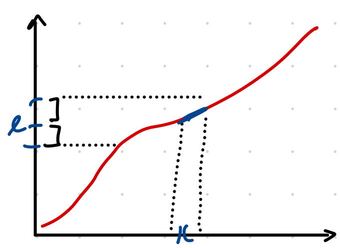
<u>Def</u>: A limit of f at c is a number $\ell \in \mathbb{R}$ s.t. $\rightarrow$ for any $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in A$ $(\neq c)$ and $x \in V_\delta(c)$ then $f(x) \in V_\varepsilon(\ell)$

$\rightarrow$ Our goal is to say "$f(x)$ is near ℓ if x is sufficiently near c"

$\rightarrow f(x) \in V_\varepsilon(\ell) \rightarrow$ "in the nbhd of ℓ" $(\ell - \varepsilon, \ell + \varepsilon)$

i.e. $|f(x) - \ell| < \varepsilon$

<u>Equiv</u>: ℓ is a limit of f at c if $\forall$ nbhd $\ell \in U$, $\exists$ a nbhd W where $c \in W$ s.t. $f(A \cap W - \{c\}) \subset U$



$\rightarrow$ A <u>nbhd</u> of $m \in \mathbb{R}$ is any open interval containing m

$\rightarrow$ A nbhd may or may not be centred at m.

To tell me of these, I could tell m ; radius θ

$V_\theta(m) := (m - \theta, m + \theta)$

What does the definition of limit say if $c \notin \text{cluster}(A)$?

$\rightarrow$ well if $c \notin \text{cluster}(A)$ then $\exists \delta$ s.t.

$$\{x \text{ s.t. } x \in A, x \neq c, x \in V_\delta(c)\} = \emptyset$$

use this δ for every ℓ, ε

we discover that <u>every</u> $\ell \in \mathbb{R}$ is a limit of f at c <u>Disaster!</u>

This does not match the qualitative idea of "limiting behaviour" so I insist $c \in \text{cluster}(A)$

Theorem: Same set-up: $A \subset \mathbb{R}$, $c \in \mathrm{Cluster}(A)$ $f: A - \{c\} \to \mathbb{R}$ if a limit of f at c exists, then it is unique.
i.e. if ℓ and ℓ' are both limits of f at c then $\ell = \ell'$

Proof: Assume for contradiction that ℓ, ℓ' are both limits of f at c and $\ell \neq \ell'$
Then $|\ell - \ell'| > 0$ (*)
$\to$ set $\varepsilon = \frac{|\ell - \ell'|}{2}$

- $\because \ell$ is a limit of f at c, $\exists \delta$ s.t. for every $x \in A \cap V_\delta(c) - \{c\}$ $|f(x) - \ell| < \varepsilon$
- $\because \ell'$ . . . ., $\exists \delta'$ - - - - - - - $A \cap V_{\delta'}(c) - \{c\}$ $|f(x) - \ell'| < \varepsilon$

$\to$ we set $\delta'' := \inf(\delta, \delta')$ then $\forall x \in A \cap V_{\delta''}(c) - \{c\}$ both inequalities.
$\because c \in \mathrm{cluster}(A)$, this set is not empty
So can pick some x in this set
For that x, the boxed inequalities (*) and $|\ell - \ell'| = 2\varepsilon$ violate triangle inequality
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Notation: Given A, c, f . . . as in set-up above write

"$\lim_{x \to \infty} f(x) = \ell$" to mean the limit exists and is equal to ℓ.
i.e. that ℓ is a limit (hence the only limit).

E.g.: Suppose $f: A \to \mathbb{R}$ is a constant function.
e.g. $f(x) = 4$ for every $x \in A$ then claim $\lim_{x \to c} 4 = 4$

Proof: i.e. Im claiming that $\forall \varepsilon > 0$, $\exists \delta$ s.t. if $|x - c| < \delta$ and then $|f(x) - 4| < \varepsilon$
The conclusion is true for every δ in particular, true when $\delta = 2$

E.g: The identity inclusion $A \hookrightarrow \mathbb{R}$ $f(x) = x$ $\lim_{x \to c} x = c$ hint: Set $\delta = \varepsilon$