Lecture 2: Set Theory Cont.

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Some features of the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$

1). contains zero ($0 \in \mathbb{N}$)
2). If $x \in \mathbb{N}$ then $x+1 \in \mathbb{N}$ also
3). $\mathbb{N}$ is the minimal set satisfying 1). & 2). $\to$ if X is a set enjoying 1). & 2). then $X \supseteq \mathbb{N}$

An aside

$\to$ Can model $\mathbb{N}$ in the language of set theory.
- $\hookrightarrow$ So lets model 0 as the empty set $\emptyset$ $\quad 0 \longleftrightarrow \emptyset = \{\}$
- $\hookrightarrow$ Can also model 1 as the all the elements we already have $1 \longleftrightarrow \{\emptyset\}$
- $\hookrightarrow$ or $2 \longleftrightarrow \{\emptyset, \{\emptyset\}\}$ $\quad 3 \longleftrightarrow \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ $\to$ its like thinking of numbers as the # of elements in the set

we can then say $n+1 \longleftrightarrow \{\text{all actors} \leq n\}$

3). let $\phi(\_)$ be a property that any given number might or might not have. (just a place holder)
thus $\phi(n) \longrightarrow$ "n has property ϕ"

example: $\phi =$ "is even" Suppose $\phi(0)$ is true and suppose $\phi(n) \Longrightarrow \phi(n+1)$
then $\phi(n)$ is true for every $n \in \mathbb{N}$

Proof: consider $\Phi = \{n \in \mathbb{N} \text{ s.t. } \phi(n)\}$ $\quad \Phi \subseteq \mathbb{N} \Longrightarrow \Phi$ satisfies 3). $\Longrightarrow \Phi \supseteq \mathbb{N}$ $\therefore \Phi = \mathbb{N}$

well ordering principle

4). Suppose $X \subseteq \mathbb{N}$ and $X \neq \emptyset$ then X has a smallest element

Method of infinite descent

Theorem: Suppose $\phi(\_)$ is a property s.t. if $\phi(x)$ is false (i.e. a counterexamp) then $\exists\, y < x$ s.t. $\phi(y)$ is false (note $x, y \in \mathbb{N}$) then $\phi(n)$ is true $\forall\, n \in \mathbb{N}$

Proof: $\Phi' := \{n \in \mathbb{N} \text{ s.t. } \neg\phi(n)\} \subseteq \mathbb{N}$ does not have a smallest element so must be $\emptyset$ by well ordered principle (WOP)

$\to$ Given $n \in \mathbb{N}$ consider the set $\mathbb{N}_{<n} \subseteq \mathbb{N}$ ($= \{0, 1, 2, \cdots, n-1\}$) $\to$ the idea is that this set has exactly n elements

$\to$ Recall from last time $f: X \to Y$ is bijective if $\forall\, y \in Y\ \exists!\, x \in X$ s.t. $f(x) = y$ AND that any bijection has an inverse defined by $f^{-1}(y) = x$ (unique x)
$f \circ f^{-1} = id_Y \quad f^{-1} \circ f = id_X$

Lemma: if f is invertible then it is bijective

$\to$ if $f: X \to Y$ & $g: Y \to Z$ are bijective then $g f: X \to Z$ is a bijection

proof:
$(gf) \circ (f^{-1} g^{-1}) = id_Z$
$(f^{-1} g^{-1}) \circ (gf) = id_X$
$f^{-1} g^{-1} g f = f^{-1} f$

$\to$ Consider the collection of all sets define symbol $\underbrace{X \cong Y}_{|X| = |Y|}$ to mean $\exists$ bijection $X \to Y$

Definition: $\underbrace{X \text{ is finite}}_{|X| < \infty}$ if $\exists\, n$ s.t. $X \cong \mathbb{N}_{<n}$ else it is infinite. $\to$ X is countable if it is finite and $\cong \mathbb{N}$

Property: if $|X| < \infty$ then there is exactly one $n \in \mathbb{N}$ s.t. $X \cong \mathbb{N}_{<n}$ in this case write $|X| = n$

→ In other words, we want to rule out the possibility that $X \cong \mathbb{N}_{<n}$ and $X \cong \mathbb{N}_{<m}$ and that $m \neq n$

→ Suffices to show that if $\mathbb{N}_{<n} \cong \mathbb{N}_{<m}$ then $n = m$ → We'll prove a theorem that will imply this

Theorem: (Pigeon hole principle)
Suppose $m > n$ $m, n \in \mathbb{N}$ then any $f : \mathbb{N}_{<m} \to \mathbb{N}_{<n}$ is not injective → equiv: if $\exists$ injective map $\mathbb{N}_{<m} \to \mathbb{N}_{<n}$ then $m \leq n$

Proof: induction in n base case: $n = 0$ then $\mathbb{N}_{<0} = \emptyset$ if $\exists f : X \to \emptyset$ then $X \neq \emptyset$ and if $m > 0$ then $\mathbb{N}_{<m} \neq \emptyset$ because it contained $m-1$.

TBC