Lecture 3: Finishing off Sets; The Real Line

12 · Jan · 2026

Finishing proof from last: Pigeonhole principle (thm) $\qquad \mathbb{N}_{<n} := \{0, 1, \dots, n-1\} = \{x \in \mathbb{N} \mid x < n\}$

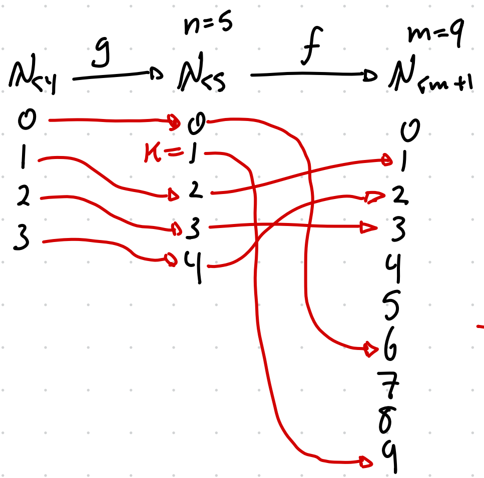
If $\exists$ an injection $f: \mathbb{N}_{<n} \to \mathbb{N}_{<m}$ then $n \le m$
$(\mathbb{N}_{<n} \hookrightarrow \mathbb{N}_{<m})$

→ Contrapositive statement: if $n > m$ then every function $f: \mathbb{N}_{<n} \to \mathbb{N}_{<m}$

Proof: Induct in m → base case: if $m = 0$ then $\mathbb{N}_{<0} = \emptyset$
→ if $X \neq \emptyset$ then there are no functions $X \to \emptyset$
→ So $\exists$ any function $\mathbb{N}_{<n} \to \mathbb{N}_{<0} = \emptyset$ then $\mathbb{N}_{<m} = \emptyset$
→ If $n \neq 0$ then $n-1 \in \mathbb{N}_{<n}$ and hence is not empty, So $n = 0$ So $n = m$

Induction step: assume this is true up to m, try to prove $m+1$ case
→ suppose given an injection $f: \mathbb{N}_{<n} \to \mathbb{N}_{<m+1}$ $= \{0, 1, \dots, m\}$
→ If m is not in the range of f then f factors through the Inclusion $\mathbb{N}_{<n} \subset \mathbb{N}_{<m+1}$
So I actually have an injection $\mathbb{N}_{<n} \hookrightarrow \mathbb{N}_{<m}$ hence $n \le m \le m+1$

→ If m is in the range of f, then pick some $k \in \mathbb{N}_{<n}$ s.t. $f(k) = m$
Define $g: \mathbb{N}_{<n-1} \to \mathbb{N}_{<n}$
$$g(x) = \begin{cases} x & x < k \\ x+1 & x > k \end{cases}$$



→ now the composite $f \circ g$ is an injection with domain $\mathbb{N}_{<n-1}$ and range some subset of $\mathbb{N}_{<m}$ so by induction $n-1 \le m$
→ So $n \le m+1$ which is what we wanted to prove ■

→ we implicitly used the fact that:

① if $f: A \to B$ and $g: B \to C$ are injections then $g \circ f$ is an injection

② if $g \circ f$ is an injection then f is an injection.

Cor: If $m \neq n$ then $\mathbb{N}_{<m} \not\cong \mathbb{N}_{<n}$
Cor: If X is finite then there is only one $n \in \mathbb{N}$ where $X \cong \mathbb{N}_{<n}$ call this $n = |X|$
Cor: $\mathbb{N}$ is not finite

Proof: Suppose conversely $\mathbb{N}$ is finite. by definition this means $\exists m$ s.t. $\exists$ a bijective map $f: \mathbb{N} \xrightarrow{\sim} \mathbb{N}_{<m}$ "bijective function"

$\xrightarrow{\sim}$ bijective map
$\hookrightarrow$ injective map
$\twoheadrightarrow$ surjective map

but the composite

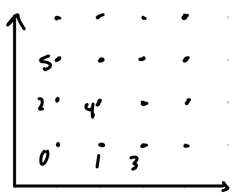
$$\mathbb{N}_{<m+1} \subset \mathbb{N} \xrightarrow{\sim} \mathbb{N}_{<m}$$ would be an injection

Definition: A set is countable if it is either finite (in bijection with $\mathbb{N}_{<m}$) or in bijection with $\mathbb{N}$ (countably finite)

Example: $\mathbb{Z}$ is countable. example $f: \mathbb{Z} \to \mathbb{N}$

$$z \longmapsto \begin{cases} 2z & \text{if } z \geq 0 \\ -2z+1 & \text{if } z < 0 \end{cases}$$

Example: $\mathbb{N} \times \mathbb{N}$



Example: $\mathbb{Q}$ = rational numbers

Theorem: Cantor
i) $\mathbb{R}$ is not countable
ii) $P(\mathbb{N}) = \{\text{Subsets of } \mathbb{N}\}$ is not countable

i)
Proof: Suppose given some $f: \mathbb{N} \to \mathbb{R}$ → Want to show it is not a bijection: Strat: look at the decimal expansion of the values of f $r(n)$

$r(0) = r(0)_n \cdots r(0)_1 r(0)_0 \,.\, r(0)_{-1} r(0)_{-2} r(0)_{-3} \cdots$
$r(1) = \cdots\cdots . r(0)_{-1} r(0)_{-2} r(0)_{-3} \cdots$
$r(2) = \cdots\cdots$
$r(3) = \cdots\cdots$

→ we'll use the decimal expansions to construct a number that is not on the list.

"needs fixing"

Now look at the number x defined by $x = 0.x_{-1}x_{-2}x_{-3}\cdots$ $\quad x_{-k} := \begin{cases} 7 & \text{if } r(k+1)_{-k} \neq 7 \\ 2 & \text{if } r(k)_{-k} = 7 \end{cases}$

get a real number x

$x \neq r(0)$ → b/c their first digit after decimal point is different
$x \neq r(1)$ → " " 2nd " " " " " "
$x \neq r(2)$ → . - - - - - . -

$\therefore$ x is not on my list so my original function r is not a surjection so r is not a bijection.

ii)
Proof: Suppose given a function $f: \mathbb{N} \to P(\mathbb{N})$ Define $A := \{n \in \mathbb{N} \mid n \in f(n)\}$ then this A is not $f(n)$ for any n.