Lecture 7: Bounds

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Last time: $\mathbb{R}$ is an ordered field and so is $\mathbb{Q}$

→ a subset S of an ordered set P e.g. $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$ is:

1). Bounded below: if $\exists\, t \in P$ s.t. $\forall s \in S$ $t \leq s$ → any such t is a lower bound
2). Bounded above: if $\exists\, t \in P$ s.t. $\forall s \in S$ $t \geq s$ → any such t is a upper bound
3). Bounded: if it is both bounded below & above.

Trivial Examples: * if $|S| = 1$ then its unique element is both lower and upper bound

* If $S = \emptyset$ then any t is both a lower and upper bound.
→ $\mathbb{N}$ is bounded below in $\mathbb{N}$
→ $\mathbb{N}$ is not bounded above in $\mathbb{N}$

→ The goal for today is to show that $\mathbb{N}$ is not bounded $\mathbb{R}$.
→ WOP (want to prove): any non-empty subset of $\mathbb{N}$ has a least element

Theorem: If $S \subset \mathbb{Z}$ is bounded below in $\mathbb{Z}$ then S has a least element
Proof: unpacking definitions, S is bounded below if $\exists t \in \mathbb{Z}$ s.t. $\forall s \in S$ $s \geq t$ → pick such a t
Define set $S - t := \{s - t \mid s \in S\}$ then $S - t \subseteq \mathbb{N}$ → so $S - t$ has a least element call it ℓ
this is positive $\because s \geq t$
then $\ell + t$ will be the least element of S

Corollary: Bounded and non-empty subsets of $\mathbb{Z}$ have both a least and greatest element

Example: $[0,1] := \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ → has a least element 0
has a greatest element 1

Example: $(0,1) := \{x \in \mathbb{R} \mid 0 < x < 1\}$ → is bounded below (by 0)
is bounded above (by 1)
"minimum" "maximum"
But has no least or greatest element:
if $x \in (0,1)$ then $\frac{x}{2} < x$ is also in $(0,1)$
$1 - \frac{1}{2}(1-x) > x$ is also in $(0,1)$

Definition: suppose $S \subseteq \mathbb{R}$ is non-empty, the <u>supremum</u> of S, if it exists, is the least upper bound of S → $\sup(S)$

The infimum ... ... is the greatest lower bound → $\inf(S)$
i.e. $\forall s \in S$, $\sup(S) \geq s$ and if $t \in \mathbb{R}$ is s.t. $\forall s \in S$ $t \geq s$ then $t \geq \sup(S)$

Example: $\sup((0,1)) = 1$

Proposition: $\sup(a,b) = b$

proof: The definition of (a,b) certainly makes b an upper bound.
→ What we to show is that if c is an upper bound of (a,b) then $c \geq b$

Want to rule out that $c < b$ $\because$ then it would not be an upper bound

Two cases: $c \geq \frac{a+b}{2}$ then $c < \frac{c+b}{2} \in (a,b)$ → so c is not an upper bound

$c < \frac{a+b}{2} \in (a,b)$ so c is not an upper bound

Axioms for $\mathbb{R}$:
- * $\mathbb{R}$ is a field
- * $\mathbb{R}$ is an ordered field
- * $\mathbb{R}$ is complete: every bounded-above subset of $\mathbb{R}$ has a supremum

→ $\mathbb{Q}$ violates completeness: $\{q \in \mathbb{Q} \mid q^2 \leq 2\}$
This is bounded above (by 2) but if it had a supremum then $(\sup(S))^2 = 2$

Theorem: Archimedean Principle

$\mathbb{N} \subset \mathbb{R}$ is not bounded above → i.e. $\nexists r \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n \leq r$
i.e. $\forall r \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n > r$

Proof: if $\mathbb{N}$ were bounded above then let $\omega := \sup(\mathbb{N})$
$\omega - 1 < \omega$ so $\omega - 1$ must not be an upper bound of $\mathbb{N}$
i.e. $\exists n \in \mathbb{N}$ s.t. $n > \omega - 1$
but then $n + 1 > \omega$

Theorem: every nonempty interval contains a rational number $\mathbb{Q}$