Lecture 8: Sequences & Convergent Sequences

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$S_n = \{0,1,2,3,4,\dots\}$

$t_n = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ → i.e. $t_n = \begin{cases} 0, \text{ if } n=0 \\ 1/n \text{ if } n>0 \end{cases}$

$f_n = \{0,1,1,2,3,5,\dots\}$ → we can also rewrite this as: $f_n = \begin{cases} 1, & n=1 \\ 0, & n=0 \\ f_{n-1}+f_{n-2} & \geq 2 \end{cases}$ "recursive definition"

$P_n = \{2,3,5,7,\dots\}$ → sequence of prime numbers

→ we should be exact when we write our sequences/sets i.e. { " " | " " " }

Definition: A sequence is a function (map) $\mathbb{N} \xrightarrow{S} \mathbb{R}$ we write $S_n := S(n)$

$0 \mapsto S(0) = S_0$

$1 \mapsto S(1) = S_1$

$\vdots \qquad \vdots$

$n \mapsto S(n) = S_n$

Example: $U: \mathbb{N} \to \mathbb{R}$

$u_n = \frac{1}{2^n}$ → $1, \frac{1}{2}, \frac{1}{4}, \dots$

$U_n = \begin{cases} 1, & n=0 \\ \frac{1}{2}U_{n-1}, & n>0 \end{cases}$

Convergent sequences: idea think back to calculus convergent series

Definition: S_n a sequence $S_n \to S$ if it satisfies the following: $\forall \varepsilon>0, \exists N$ s.t. for every $n \geq N$

→ S_n converges to S

$\lim_{n\to\infty} S_n = S$

$\Rightarrow |S_n - S| < \varepsilon \iff -\varepsilon < S_n - S < \varepsilon$

$\iff -\varepsilon + S < S_n < \varepsilon + S$

→ ε is arbitrarily as small as you want

Example: $b_n = b$ (constant sequence) **claim:** $b_n \to b$

$\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $n \geq N$ $|b_n - b| < \varepsilon$ → but $b_n = b$

$0 = |b - b|$

say $N = 17$ $n \geq 17$

$|b_n - b| < \varepsilon$

Example: $\frac{1}{n} \to 0$ $t_n = \frac{1}{n}$ $\forall \varepsilon>0, \exists N$ s.t. $n \geq N$ → $|t_n - 0| < \varepsilon$

need to see if this exists

$|\frac{1}{n} - 0|$

$\forall \varepsilon > 0 \ \exists ? N$ s.t. $n \geq N$ $\frac{1}{n} < \varepsilon$

$\frac{1}{\varepsilon} < n$

→ $\exists N$ s.t. $\frac{1}{\varepsilon} < N$

$n \geq N \Rightarrow n > \frac{1}{\varepsilon}$

Example: $\frac{1}{\varepsilon} = \frac{1}{100^{100}}$ and have $N = \frac{1}{\varepsilon} + 1 = 100^{100} + 1$ $n \geq N \to n \geq 100^{100} + 1 > \frac{1}{\varepsilon} \to \varepsilon > \frac{1}{n}$

Example: $\frac{3n+2}{5n-1} \to \frac{3}{5}$ again our task is to find $\forall \varepsilon>0, \exists ? N$ s.t. $n \geq N \to \left|\frac{3n+2}{5n-1} - \frac{3}{5}\right| < \varepsilon$

→ $\left|\frac{5n+10-15n-3}{25n+5}\right| = \frac{7}{25n+5} < \varepsilon$ → $\frac{7}{25n+5} < \frac{7}{25n}$

Fact: $\exists N_\varepsilon$ s.t. $\frac{7}{25N} < \varepsilon$ $n \geq N_\varepsilon \Rightarrow \frac{7}{25n} \leq \frac{7}{25N} < \varepsilon$

$|S_n - S|$

Exercise: $\underbrace{\frac{1}{\sqrt{n}}}_{=s_n} \longrightarrow \underbrace{0}_{=s}$ $\longrightarrow$ let $\varepsilon>0$ then we must find N_ε s.t. $n\geq N_\varepsilon \implies |s_n - s|<\varepsilon$

$n\geq N \Rightarrow \left|\frac{1}{\sqrt{n}}-0\right|<\varepsilon \longrightarrow \frac{1}{\sqrt{n}}<\varepsilon$

$\frac{1}{\varepsilon^2}<n$

eg. $N=\frac{1}{\varepsilon^2}+17$

$n\geq N \implies n\geq \frac{1}{\varepsilon^2}+17>\frac{1}{\varepsilon^2}$

suppose $\varepsilon=5^{-10}=\frac{1}{5^{10}}$ $\quad n\geq N \implies \frac{1}{\varepsilon^2} \Longleftrightarrow 5^{20}<n$

$\longrightarrow$ e.g. $N=6^{20}$ indeed $n\geq 6^{20}>5^{20}$

$N=5^{20}+1$ indeed $n\geq 5^{20}+1>5^{20}$ also works for

Example: $q_n=(-1)^n=\{1,-1,1,-1,\cdots\}$ $\quad q_n \not\to 0$ $\quad \forall\, \varepsilon>0,\ \exists N\ (n\geq N \implies \underbrace{|q_n-0|}_{=1}<\varepsilon$

$1<\varepsilon$ not true

Proposition: x_n converges to at most one number

proof: $x_n \longrightarrow x'$

$x_n \longrightarrow x''$

$\begin{cases} \forall\varepsilon,\ N',\ n\geq N' \implies |x_n - x'|<\varepsilon \\ \forall\varepsilon,\ N'',\ n\geq N'' \implies |x_n - x''|<\varepsilon \end{cases}$

$n\geq \max(N',N'') \implies |x_n-x'|+|x''-x_n|<2\varepsilon$

$|x_n-x'+x''-x_n|<2\varepsilon$

$\forall\varepsilon,\ |x''-x'|<2\varepsilon \implies x''-x'=0$

$x''=x'$ $\blacksquare$