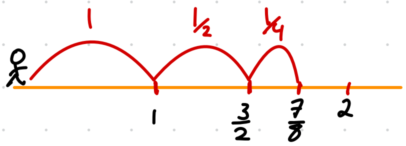
Lecture 9: Sequences & Series Cont.

30.Jan.2026

Warmup: Let $S_n = \sum_{K=1}^{n} \frac{1}{2^K} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \longrightarrow$ What does this converge to? lets guess 2



given $\varepsilon > 0$ we need N s.t. $n \geq N$, then $|S_n - S| < \varepsilon$

$$|S_n - S| = \left|\left(\sum_{K=0}^{n} \frac{1}{2^K}\right) - 2\right| = \left|\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} - 2\right| = \left|\cancel{2} - \frac{1}{2^{n+1}} - \cancel{2}\right| = \frac{1}{2^{n+1}}$$

$a_n = a_0 r^K$

$$\sum_{K=0}^{n} a_K = a_0 \frac{1 - r^{n+1}}{1 - r^K}$$

Fact: $\exists N$ s.t. $\frac{1}{2^N} < \varepsilon$ $\quad n \geq N \Rightarrow n+1 \geq N+1$

$$= \frac{1}{2^{n+1}} \leq \frac{1}{2^N} < \varepsilon$$

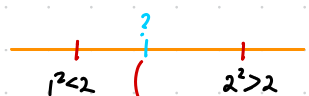
$\therefore S_n \longrightarrow 2$ ∎

$\sqrt{2} := t$ for $t \geq 0$ s.t. $t^2 = 2 \longrightarrow$ how do we show such a t exists?

Recall: that a bounded set $B \subseteq \mathbb{R}$ has a supremum

$N = \{x \mid x^2 \leq 2\} \longrightarrow \sqrt{2} = \sup(N)$

$\longrightarrow$ We can also represent $\sqrt{2}$ as a sequence



$$(a_n, b_n) = \begin{cases} (1,2] & n=0 \\ (a_n, \text{avg}) & \left(\frac{a_n + b_n}{2}\right)^2 > 2 \\ (\text{avg}, b_n) \end{cases}$$

$(1,2)$
$(1, 1.5)$
$(1.25, 1.5)$
$\vdots$
(a_n, b_n)

Assignment 1: Q4 $P_{fin}(\mathbb{N})$ = finite subsets of $\mathbb{N}$ is countable

P_n = set of all finite subsets of size n $\quad P_{fin}(\mathbb{N}) = \bigcup_{i=1}^{\infty} P_n$

1. P_n is countable
2. Countable union of countable sets is countable $\longrightarrow B = \bigcup_{i=0}^{\infty} B_i$ $\quad B_i \xhookrightarrow{f_i} \mathbb{N}$

   $A \hookrightarrow \mathbb{N}$ injective $\quad B \hookrightarrow \mathbb{N} \times \mathbb{N}$

3. $\mathbb{N} \times \mathbb{N}$ is countable

$x \in B_n \longmapsto (n, f_n(x))$

n as large as possible

Definition: S_n is bounded if $\exists M \in \mathbb{R}$ s.t. $|S_n| < M$

Theorem: if S_n converges to S then it is bounded

Proof: $\varepsilon = 1$ $\quad \exists M$ s.t. $|S_n - S| < 1$ if $n \geq N \longrightarrow |S_n| = |S_n - S + S| \leq |S_n - S| + |S| = 1 + |S| < 2 + |S|$

$$M = \sup(2 + |S|, |S_1|, |S_2|, \ldots, |S_{N-1}|)$$

Example: $t_n = n\,(1, 2, 3, 4, \ldots)$ not convergent

Proposition: $S_n \longrightarrow S$, $t_n \longrightarrow t$ $\longrightarrow$

a. $S_n + t_n \longrightarrow S + t$
b. $S_n - t_n \longrightarrow S - t$
c. $S_n \cdot t_n \longrightarrow S \cdot t$
d. if $t \neq 0$, $S_n / t_n \longrightarrow S/t$

Remark: $t_n = n$, $-t_n = -n \longrightarrow t_n - t_n = 0$

Proof: a. have: $\begin{cases} \forall \varepsilon, \exists N \text{ s.t. } |S_n - S| < \varepsilon \\ \forall \varepsilon, \exists N \text{ s.t. } |t_n - t| < \varepsilon \end{cases}$ $\quad n \geq N$

Want: $\forall \phi, \exists M$ s.t. $|S_n + t_n - (S + t)| < \phi$

$\longrightarrow$ For given ϕ, need to find M

$\forall \varepsilon, N$ s.t. $n \geq N$

$|S_n - S| + |t_n - t| < 2\varepsilon$ $\longrightarrow$ triangle inequality $\quad |S_n - S + t_n - t| < 2\varepsilon$

$|S_n + t_n - (S + t)| < 2\varepsilon \longrightarrow n \geq N$

take $\varepsilon = \frac{1}{2}\phi$

$|S_n + t_n - (S + t)| < 2 \cdot \frac{1}{2}\phi = \phi$

Proposition: $X_n \to L$
(squeeze theorem) $y_n \to L \implies Z_n \to L \quad \to x_n \leq Z_n \leq y_n \quad x=y$

$-\varepsilon < |Z_n - L| < \varepsilon$

Proof: wts $\forall \varepsilon, N$ s.t. $n \geq N$ $(|Z_n - L| < \varepsilon)$

we have $\forall \varepsilon, \exists N$ s.t. $n \geq N$ $|x_n - L| < \varepsilon$ $\quad -\varepsilon < x_n - L \leq Z_n - L \leq y_n - L < \varepsilon$

$|y_n - L| < \varepsilon$