Lecture 15: Sequential limit Theorem

27. Feb. 2026

"Sequential limit Theorem"

Theorem: Suppose $A \subset \mathbb{R}$, c is a cluster point of A, and $f: A-\{c\} \to \mathbb{R}$ is a function. Then TFAE:

1). $\lim_{x\to c} f(x) = \ell$

2). for every sequence x_n in $A-\{c\}$ s.t. $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = \ell$

Proof: ((1) $\Rightarrow$ (2)) Assuming condition (1) and first half of (2)

(*) content of (1) By definition, $\forall \varepsilon>0, \exists \delta$ s.t. if $x \in (A-\{c\} \cap V_\delta(c))$ then $f(x) \in V_\varepsilon(\ell)$

(**) $\to$ Saying $x_n \to c$ is by definition saying $\forall \theta \, \exists N$ s.t. if $n \geq N$ then $x_n \in V_\theta(c)$

WTS: Given above then $\lim_{n\to\infty} f(x_n) = \ell$ $\longrightarrow$ i.e. WTS: $\forall \varepsilon>0, \exists M$ s.t. if $n>M$ then $f(x_n) \in V_\varepsilon(\ell)$

Given ε, find δ using (*), then set $\theta = \delta$, find N using (**) then set $M=N$, win.

Very Explicitly: to see that this wins, I need to check:

$\to$ If $n > N(\delta(\varepsilon))$ then $f(x_n) \in V_\varepsilon(\ell)$

$\to$ If $n > N(\delta(\varepsilon))$ then $x_n \in V_{\delta(\varepsilon)}(c)$ (**)

thus $f(x_n) \in V_\varepsilon(\ell)$ (*)

Proof: ((2) $\Rightarrow$ (1)) Assume condition (1) fails, i.e.: $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, you can find an $x \in A-\{c\} \cap V_\delta(c)$ for which $f(x) \notin V_\varepsilon(\ell)$

$\to$ Fix this ε and set $\delta = \frac{1}{n}$

$\to$ Pick an $x_n \in A-\{c\} \cap V_{1/n}(c)$ with $f(x_n) \notin V_\varepsilon(\ell)$

This sequence x_n does converge to c

Claim: For this sequence x_n, $\lim_{n\to\infty} f(x_n) \neq \ell$

In fact, for this sequence, $f(x_n)$ is never close to ℓ.

So it either dosnt have a limit or its limit is far from ℓ.

! in more general settings, (not subsets of $\mathbb{R}$) (1) $\Longrightarrow$ (2) but (2) $\not\Longrightarrow$ (1)

Corollary: Same setup: TFAE

i) $\lim_{x\to c} f(x)$ Does not exist (DNE)

ii) $\exists$ sequence x_n in $A-\{c\}$ with $\lim_{n\to\infty} x_n = c$ s.t. $\lim_{n\to\infty} f(x_n)$ DNE

Proof ii) $\Rightarrow$ i) immediate; if $\lim_{x\to c} f(x)$ exists, then every sequence converges to it.

Proof i) $\Rightarrow$ ii) The sequencial limit theorem said: $\lim_{x\to c} f(x)$ exists $\Longleftrightarrow \forall x_n \to c$ $\lim f(x_n)$ exists and that all these limits are the same.

→ So if (i), then either:

a) Some $x_n \to c$ has limit $f(x_n)$ DNE (what we want to show)

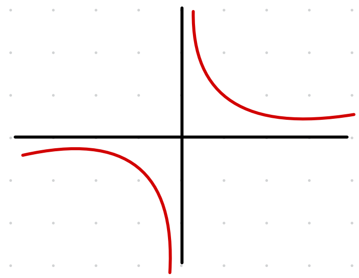
b) there are sequences $x_n \to c$ and $y_n \to c$ s.t.

$\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ both exist but are different.

In case (b), I'll set $z_n := \begin{cases} x_n & \text{if } n \text{ even} \\ y_n & \text{if } n \text{ odd} \end{cases}$ then $\lim_{n\to\infty} z_n = c$

But $\lim_{n\to\infty} f(z_n)$ DNE ■

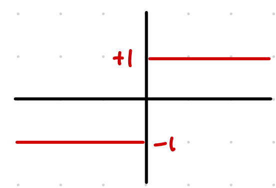
Example: $A = \mathbb{R} - \{0\}$ and $f(x) = \frac{1}{x}$ $\lim_{x\to 0} f(x)$ DNE



Proof: look at the sequence $x_n = \frac{1}{n}$ then we have

$\frac{1}{x_n} = n$ and $\lim_{n\to\infty} n$ DNE

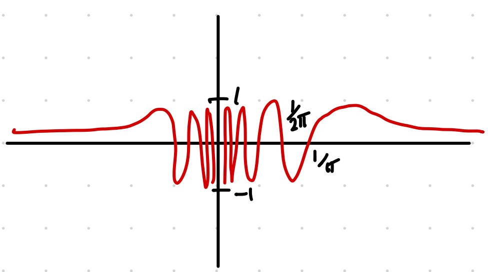
Example: $A = \mathbb{R} - \{0\}$ $\frac{x}{|x|} := \text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$



Claim: $\lim_{x\to 0} \text{sign}(x)$ DNE

proof: take the sequence $x_n = (-1)^n \frac{1}{n}$ then $\text{sign}(x_n) = (-1)^n$ $\lim_{n\to\infty} (-1)^n$ DNE

Example: $A = \mathbb{R} - \{0\}$ $f(x) = \sin\left(\frac{1}{x}\right)$ set $x_n = \frac{1}{(n+\frac{1}{2})\pi}$ $\sin\left(\frac{1}{x_n}\right) = \sin\left(\left(n+\frac{1}{2}\right)\pi\right) = (-1)^n$



Values oscillate b/w 1 : -1

for this x_n $\lim f(x_n)$ DNE