Lecture 16: limits of Functions Cont.

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The set-up: $A \subset \mathbb{R}$ $\quad$ c is a cluster point of A by $A \rightsquigarrow A - \{c\}$

$f: A \to \mathbb{R}$ $\quad$ say: $\lim_{x\to} f = \ell \longrightarrow$ means $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $x \in A \cap V_\delta(c)$ then $f(x) \in V_\varepsilon(\ell)$

Recall: Can happen that the limit DNE
if such ℓ exists, then it is unique $\longrightarrow$ i.e. $\{f: A \to \mathbb{R}\} \supseteq \{f \text{ s.t. } \lim_{x\to c} f \text{ exists}\} \xrightarrow[\lim_{x\to c}]{} \mathbb{R}$

Slogan of today: "$\lim_{x\to c}$ of some function $\to \mathbb{R}$ respects basic operations"

$\to$ Given $f \cdot g: A \to \mathbb{R}$ $\quad$ define $f+g: A \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$

Theorem: if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}(f+g)(x)$ exists and

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

! Set $A = \mathbb{R} - 0$ $\quad$ $f = \text{sign}$ $\quad$ $g = -\text{sign} \longrightarrow$ then $f+g = 0$ so LHS exists but RHS DNE

Proof: wts that $\forall \varepsilon > 0\ \exists \delta_{f+g}(\varepsilon) > 0$ s.t. if $x \in A \cap V_{\delta_{f+g}(\varepsilon)}(c)$ then $(f+g)(x) \in V_\varepsilon(\ell_f + \ell_g)$

Set $\ell_f = \lim_{x\to c} f(x)$ $\quad \longrightarrow$ Know: $\cdot\ \forall \varepsilon > 0\ \exists \delta_f(\varepsilon) > 0$ s.t. if $x \in A \cap V_{\delta_f}(c)$ then $f(x) \in V_\varepsilon(\ell_f)$
$\cdot\ \forall \varepsilon > 0\ \exists \delta_g(\varepsilon) > 0$ s.t. if $x \in A \cap V_{\delta_g}(c)$ then $g(x) \in V_\varepsilon(\ell_g)$

$\ell_g = \lim_{x\to c} g(x)$

$\longrightarrow$ "$a \overset{\varepsilon}{\approx} b$" $\ |a-b| < \varepsilon$ $\quad$ Know: if $x \overset{\delta_f}{\approx} c$ and $x \overset{\delta_g}{\approx} c$ then $f(x) \overset{\varepsilon}{\approx} \ell_f$ and $g(x) \overset{\varepsilon}{\approx} \ell_g$
$\Longrightarrow$ then $(f+g)(x) \overset{2\varepsilon}{\approx} \ell_f + \ell_g$

$\because \lim_{x\to c} f(x) = \ell_f$ $\quad$ I can find $\delta_f(\frac{\varepsilon}{2})$ s.t. if $x \in V_{\delta_f}(c)$ then $f(x) \in V_{\varepsilon/2}(\ell_f)$
Similarly $\lim_{x\to c} g(x) = \ell_g$ I can find $\delta_g(\frac{\varepsilon}{2})$ s.t. if $x \in V_{\delta_g}(c)$ then $g(x) \in V_{\varepsilon/2}(\ell_g)$ $\quad (*)$

$\longrightarrow$ Set $\delta_{f+g}(\varepsilon) = \min\left(\delta_f(\frac{\varepsilon}{2}), \delta_g(\frac{\varepsilon}{2})\right)$ $\quad$ then if $x \in V_{\delta_{f+g}}(c)$ then $x \in V_{\delta_f}(c)$ and $V_{\delta_g}(c)$

and thus $(*)$ and thus $(f+g)(x) \in V_\varepsilon(\ell_f + \ell_g)$

Now for $f \cdot g$: Given $f \cdot g: A \to \mathbb{R}$ $\quad (f\cdot g)(x) = f(x) \cdot g(x)$

Theorem: if $\lim_c f$ and $\lim_c g$ exist then $\lim_c (f\cdot g)$ exists and $\lim_c (f \cdot g) = \lim_c f \cdot \lim_c g$

$\to$ How do errors compound when multiplying? if $f(x) \overset{\varepsilon_f}{\approx} \ell_f$ and $g(x) \overset{\varepsilon_g}{\approx} \ell_g$ then what?

$$\ell_f - \varepsilon_f < f(x) < \ell_f + \varepsilon_f$$
$$\ell_g - \varepsilon_g < g(x) < \ell_g + \varepsilon_g$$
$\longrightarrow$ now we multiply the inequalities together

i) if $\ell_f, \ell_g > 0$ and $0 < \varepsilon_f < \ell_f$, $0 < \varepsilon_g < \ell_g$ $\longrightarrow$ everything above stays positive then $\ell_f\ell_g - \varepsilon_f\ell_g - \varepsilon_g\ell_f + \varepsilon_f\varepsilon_g < f(x)\cdot g(x) < \ell_f\ell_g + \varepsilon_f\ell_g + \varepsilon_g\ell_f + \varepsilon_f\varepsilon_g$

$$\Longrightarrow \quad \boxed{|f(x)\cdot g(x) - \ell_f\ell_g| < \varepsilon_f|\ell_g| + \varepsilon_g|\ell_f| + \varepsilon_f\varepsilon_g}$$

Think through ℓ_f or ℓ_g or both are zero or negative we find the boxed inequality always holds

Proof: I want total error $< \varepsilon$ so I win if I pick $\varepsilon_f, \varepsilon_g$ (depending on ε) arranging this.

$\varepsilon_f|\ell_g| + \varepsilon_g|\ell_f| + \varepsilon_f\varepsilon_g$

if $\varepsilon_f < \frac{\varepsilon}{2|\ell_g|}$ and $\varepsilon_g < \frac{\varepsilon}{2|\ell_f|}$ then $\varepsilon_f|\ell_g| + \varepsilon_g|\ell_f| < \varepsilon$ thus total error $> \varepsilon + \frac{\varepsilon^2}{4|\ell_f||\ell_g|}$

Break up into cases

i). if $\ell_f, \ell_g \neq 0$ then try $\varepsilon_f := \frac{\varepsilon}{3\ell_g}$ and $\varepsilon_g := \frac{\varepsilon}{3\ell_f}$ then $\varepsilon_f|\ell_g| + \varepsilon_g|\ell_f| + \varepsilon_f\varepsilon_g = \frac{2}{3}\varepsilon + \frac{\varepsilon^2}{9|\ell_f\ell_g|}$

this wins if $\varepsilon < 3|\ell_f\ell_g| \longrightarrow$ arranges total error $< \varepsilon$

if $\varepsilon < 3|\ell_f\ell_g|$ then set $\delta_{fg}(\varepsilon) = \min\left(\delta_f\left(\frac{\varepsilon}{3|\ell_g|}\right), \delta_g\left(\frac{\varepsilon}{3|\ell_f|}\right)\right)$; and then (some string of inequalities later) find it works

if $\varepsilon \geq 3|\ell_f\ell_g|$ use $3|\ell_f\ell_g|$ instead

ii) $\ell_f = 0$ but $\ell_g \neq 0$

iii) $\ell_f = \ell_g = 0$ $\quad \varepsilon = \sqrt{\varepsilon}^2 \longrightarrow$ so $\delta_{fg}(\varepsilon) = \min\left(\delta_f(\sqrt{\varepsilon}), \delta_g(\sqrt{\varepsilon})\right)$