Lecture 17: More results about limits

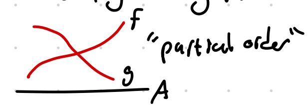
04. Mar. 2026

Set up: $A \subset \mathbb{R}$, $c \in \text{cluster}(A)$ $\quad c \notin A$

Theorem: Suppose we have functions $f, g : A \to \mathbb{R}$ s.t. $f \le g$ and that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exists then $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$

Definition: if $f, g : A \to \mathbb{R}$ say "$f \le g$" if $f(x) \le g(x)\ \forall x \in A$

$\to$ it could happen that $f \not\le g$ and $g \not\le f$



Proof: Equivalently the theorem claims that if $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$ then $\exists x \in A$ s.t. $f(x) > g(x)$

$\to$ Set $\varepsilon := \frac{1}{2}(\lim f - \lim g)$ Since $\lim f$ exist, $\exists \delta_f$ s.t. if $x \in A \cap V_{\delta_f}(c)$ then $\to$ $|f(x) - \lim f| < \varepsilon$

$\exists \delta_g$ s.t. ... $|g(x) - \lim g| < \varepsilon$ $\to$ by setting $\delta = \min(\delta_f, \delta_g)$ we find that if $x \in A \cap V_\delta(c)$

then both hold

$\because \varepsilon := \frac{1}{2}(\lim f - \lim g)$ $\qquad f(x) > \lim f - \varepsilon$

$$\lim f - \varepsilon = \lim g + \varepsilon > g(x)$$

i.e. if $x \in A \cap V_\delta(c)$ then $f(x) > g(x)$

$\because c \in \text{cluster}(A)$

$A \cap V_\delta(c) \neq \emptyset$ ∎

Squeeze Theorem: Now suppose $f \le g \le h$ where $f, g, h : A \to \mathbb{R}$

s.t. $\lim f = \lim h = \ell$ $\to$ Then $\lim g$ exists By the prev theorem, will have $\ell \le \lim g \le \ell$ so $\lim g = \ell$

Proof: Given ε, find $\delta = \min(\delta_f, \delta_h)$ so that if $x \in A \cap V_\delta(c)$ then $f(x) \in V_\varepsilon(\ell)$ and $h(x) \in V_\varepsilon(\ell)$

$\to$ In particular

$$\ell - \varepsilon < f(x) \le g(x) \le h(x) < \ell + \varepsilon \quad \text{so } g(x) \in V_\varepsilon(\ell)$$ ∎

Example: $-1 \le \sin(x^{-3} - e^{1/x} + \sqrt{x}) \le 1$

$$-|x|^{0.1} \le \underbrace{|x|^{0.1} \sin(x^{-3} - e^{1/x} + \sqrt{x})}_{} \le |x|^{0.1}$$

$\lim_{x\to 0}(\ )$ must be 0 b/c $(\ )$ is squeezed b/w $-|x|^{0.1}$ and $|x|^{0.1}$

and $\underbrace{\lim_{x\to c} \pm|x|^{0.1} = 0}_{★}$ $\qquad$ Proof of ★ Given $\varepsilon > 0$ set $\delta = \varepsilon^{10}$

use if $x < y$ then $x^{0.1} < y^{0.1}$ with $y = \varepsilon^{10}$

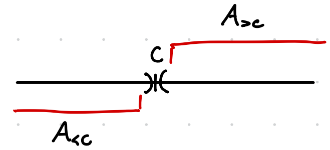
Comment on variations of "limit": "one-sided limits"

Given A, and $c \in cl(A)$ $\to$ cut $A = A_{<c} \cup A_{>c}$

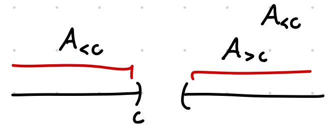
$$A_{<c} := \{x \in A \mid x < c\}$$
$$A_{>c} := \{x \in A \mid x > c\}$$

→ Definitely will happen that c is a cluster point of at least one-half

options: • $c \in cl(A_{<c})$ and $c \in cl(A_{>c})$ e.g.: $A = \mathbb{R} - \{c\}$



• $c \in cl(A_{<c})$ but $c \notin cl(A_{>c})$ e.g.:



• $c \notin cl(A_{<c})$ but $c \in cl(A_{>c})$

claim: it does not happen that

• $c \notin cl(A_{<c})$ and $c \notin cl(A_{>c})$

Proof: suppose $c \notin cl(A_{<c})$ and $c \in cl(A_{>c})$ then $\exists\, \delta_<$ s.t. $A_{<c} \cap V_{\delta_<}(c) = \emptyset$
and $\exists\, \delta_>$ s.t. $A_{>c} \cap V_{\delta_>}(c) = \emptyset$

By taking $\delta = \min(\delta_>, \delta_<)$ we get $A_{<c} \cap V_\delta(c) = \emptyset = A_{>c} \cap V_\delta(c)$

But $A = A_{<c} \cup A_{>c}$ so $A \cap V_\delta(c) = (A_{<c} \cap V_\delta(c)) \cup (A_{>c} \cap V_\delta(c))$

$= \emptyset \cap \emptyset = \emptyset$

→ this contradicts assumption that $c \in cl(A)$ $f: A \to \mathbb{R}$

→ If $c \in cl(A_{<c})$ then define "$\lim\limits_{x \to c} f$" to be the limit of $f|_{A_{>c}}$ (if it exists).

$c \in cl(A_{>c})$ then can be defined "$\lim\limits_{x \searrow c} f$"

Example: $A = \mathbb{R} - \{0\}$ $f(x) = sign(x) = \frac{x}{|x|}$ $\lim\limits_{x \to 0^+} sign = +1$ $\lim\limits_{x \to 0^-} sign = -1$ and $\lim\limits_{x \to 0} sign$ DNE

$\lim\limits_{x \to c}$ exists $\Longleftrightarrow$ both $\lim\limits_{x \to c^-}$ and $\lim\limits_{x \to c^+}$ exist and they are equal

→ $\lim\limits_{x \to c^-}$ exists $\Longleftrightarrow$ $\lim\limits_{x \to c}$ exists and they are equal $\lim\limits_{x \to c^+} = \lim\limits_{x \to c}$

Behaviour if I invite $c \in A$? → if $\forall \varepsilon > 0\ \exists\, \delta$ s.t. if $x \in V_\delta(c) \cap A$ then $f(x) \in V_\varepsilon(\ell)$

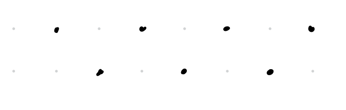
→ Now Im letting $x = c$ as an example input. $c \in V_\delta(c) \cap A$
So whatever this modified definition is, it forces $\forall \varepsilon,\ f(c) \in V_\varepsilon(\ell)$
this forces $f(c) = \ell$

Definition: Suppose $A \subset \mathbb{R}$, $c \in A$, $f: A \to \mathbb{R}$; f is continuous at c if

$$\forall \varepsilon > 0\ \exists\, \delta > 0 \text{ s.t. if } x \in A \cap V_\delta(c) \text{ then } f(x) \in V_\varepsilon(f(c))$$

Theorem: f is continuous at $c \Longleftrightarrow \lim\limits_{x \to c} f(x) = f(c)$

Crazy Example: $A = \mathbb{R}$ $f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{|b|} & \text{if } x = \frac{a}{b} \text{ is lowest term} \end{cases}$



continuous at $c \Longleftrightarrow c \notin \mathbb{Q}$