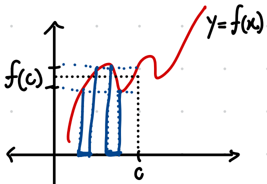
Lecture 18:

06-mar-2026

Definition: Let $A \subset \mathbb{R}$ and $f: A \to \mathbb{R}$ and $c \in A$

f is <u>continuos at c</u> if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ s.t. if $x \in A \cap V_\delta(c)$ then $f(x) \in V_\varepsilon(f(c))$

Equivalently:



for any nbhd $U \ni f(c)$ (e.g. $V_\varepsilon(f(c))$)
$\exists$ nbhd $V \ni c$ s.t. $f^{-1}(U) = \{x \in A \mid f(x) \in U\}$
$\supseteq V$

let $A \subset \mathbb{R}$; $f: A \to \mathbb{R}$; $c \in A$

Proposition: 1). if $c \notin cl(A)$ then f is cont. at c

2). if $c \in cl(A)$ then TFAE:

a). f continuous at c

b). $\lim_{x \to c} f(x)$ exists and $= f(c)$

Proof: 1). if $c \notin cl(A)$ then $\exists \delta$ s.t. $V_\delta(c) \cap A = \{c\}$, in this can can use this δ for every ε

$\to$ This works b/c if $x \in A \cap V_\delta(c)$ then $x = c$ and so $f(x) = f(c) \in V_\varepsilon(f(c))$

2) if c is a cluster point of A then the only difference b/w definition of "continous at c" and definition of "$\lim_{x \to c} f(x) = f(c)$"

is that in the latter,

"…. if $x \in A - \{c\} \cap V_\delta(c)$ …"

whereas the former,

"…. if $x \in A \cap V_\delta(c)$…."

Both definitions end with "…. then $f(x) \in V_\varepsilon(f(c))$" if $x = c$ then $f(x) = f(c) \in V_\varepsilon(f(c))$ so the exclusion of the $x = c$ so dosnt chang the test.

Corollary: If $c \in cl(A)$ then f is continuous at $c \iff \forall$ sequence x_n in A with $x_n \to c$ have $f(x_n) \to f(c)$

Proof: use the sequential theorem.

Contrapositively: f is discontinuous ("not continuous") $\iff \exists$ some $x_n \to c$ in A s.t. $\lim_{x_n \to c} f(x_n)$ either dosnt exist or exists but $\neq f(c)$

Definition: if $B \subset A$, it is <u>continuous on B</u> if f is conts at c $\forall c \in B$

in particular f is continous if it is continuous on all its domain

Lemma: Suppose $f: A \to \mathbb{R}$ $c \in A$ $c \in B \subset A$

if f is cont at c then $f|_B$ is cont. at c.

proof: …if $x \in A \cap V_\delta(c)$ then --- …if $x \in B \cap V_\delta(c)$ then…

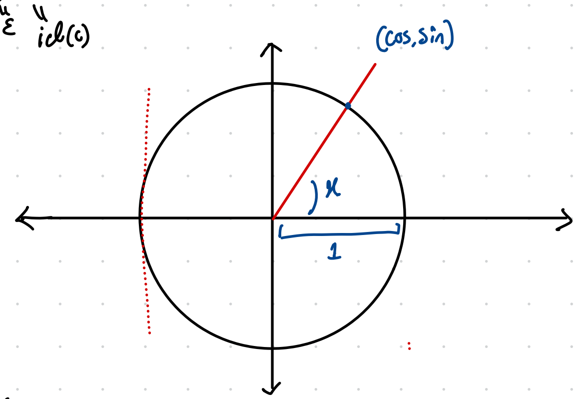
note: $B \cap V_\delta(c) \subset A \cap V_\delta(c)$

Examples: (0). Pick any $r \in \mathbb{R}$ the constant function $\text{const}_r : \mathbb{R} \to \mathbb{R}$ defined by $\text{const}_r(x) = r$ is continuous on $\mathbb{R}$

Why? ... $f(x) \in V_\varepsilon(f(c))$ this is always true $\forall x, c, \varepsilon$

(1). identity function $id : \mathbb{R} \to \mathbb{R}$ $id(x) = x$ is continuous

<u>Why?</u> set $\delta = \varepsilon$ then if $x \in V_\delta(c)$ then $\underset{id(x)}{x} \in \underset{\varepsilon}{V_\delta} \underset{id(c)}{(c)}$



(100). <u>claim:</u> $\cos : \mathbb{R} \to \mathbb{R}$, $\sin : \mathbb{R} \to \mathbb{R}$ are continuous

<u>Proof:</u> $|\cos(x)|, |\sin(x)| \leq 1$

$(\cos(x+y), \sin(x+y))$
$= (\cos(x)\cos(y) - \sin(x)\sin(y),$
$\sin(x)\cos(y) + \cos(x)\sin(y))$

→ A way to prove this:

a) show that "rotate by x" is a linear transformation

b) $\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}\cos x\\ \sin x\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}$ $\therefore$ "rotate by x" $= \begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{pmatrix}$

"rotate by $(x+y)$" = rotate by x ∘ rotate by y

$$\Longrightarrow \begin{pmatrix}\cos(x+y) & -\sin(x+y)\\ \sin(x+y) & \cos(x+y)\end{pmatrix} = \begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{pmatrix} \cdot \begin{pmatrix}\cos y & -\sin y\\ \sin y & \cos y\end{pmatrix}$$

$\cos(-y) = \cos(y)$ $\sin(-y) = -\sin(y)$ → By setting $x' = x+y$; $y' = x-y$

<u>find:</u> $\cos(x) - \cos(y) = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$ $\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x-y}{2}\right)$

<u>final fact:</u> $|\sin x| \leq |x|$ $\forall x \in \mathbb{R}$

→ Combining these gives: $\forall x, y$ $|\cos(x) - \cos(y)| = |2\sin(\ )\cdot\sin\left(\frac{x-y}{2}\right)|$

☆ $= 2\underbrace{|\sin(-)|}_{\leq 1}\cdot\underbrace{|\sin\left(\frac{x-y}{2}\right)|}_{\leq \left|\frac{x-y}{2}\right|}$

$\leq |x-y|$

☆ $|\sin(x) - \sin(y)| = 2\underbrace{|\cos(-)|}_{\leq 1}\cdot\underbrace{|\sin\left(\frac{x-y}{2}\right)|}_{\leq \left|\frac{x-y}{2}\right|}$

$\leq |x-y|$

→ I am trying to prove that cos and sin are continuous, WTS:

$\forall c, \forall \varepsilon, \exists \delta$ s.t. if $x \leftarrow V_\delta(c)$ s.t.

$|x-c| < \delta$

$\cos(x) \in V_\varepsilon(\cos(c))$ → $|\cos(x) - \cos(c)| < \varepsilon$

$\sin(x) \in V_\varepsilon(\sin(c))$ → $|\sin(x) - \sin(c)| < \varepsilon$

set $\delta = \varepsilon$ use ☆ with $y = c$