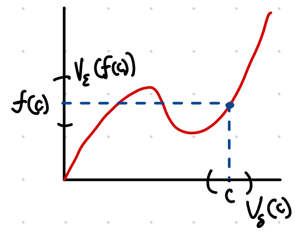
Lecture 19: Function Limits

09 · March · 2026

Recall: A function $f: \mathbb{R} \to \mathbb{R}$ is continous at $c \in \mathbb{R}$ if for any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $f(V_\delta(c)) \subset V_\varepsilon(f(c))$

encoding a small nbhd $V_\varepsilon(f(c))$

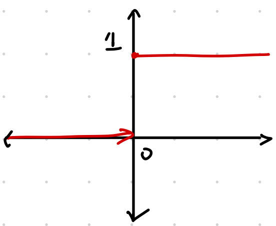
endcoding a small nbhd of c



* f is discont at c if not cont. at c
* dis(cont.) on $B \subset \mathbb{R}$ if (dis)cont. at every $c \in B$

Chesnut examples:

→ If $A \subset \mathbb{R}$ there is a function (indicator function) $ind_A: \mathbb{R} \to \mathbb{R}$ defined by $ind_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$



$ind_{[0,1]}$ is discont on $\{0,1\}$ but cont. every where else

Example: The Dirichlet function is $ind_{\mathbb{Q}}$

Claim: $ind_{\mathbb{Q}}$ is discont. everywhere. → why? $\forall c, \forall \delta$, the neighborhood $V_\varepsilon(c)$ contains both rational and irrational points

So if $\varepsilon < 1$, $f(V_\delta(c)) \not\subset V_\varepsilon(f(c))$

Proposition: Suppose $f, g: \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ s.t. f is bounded and $g(c) = 0$ and g is continuous at c.

then $f \cdot g$ is contin. at c.

$(f \cdot g)(x) = f(x) \cdot g(x)$ → $\therefore f(c) \cdot g(c) = f(c) \cdot 0 = 0$

proof: → Know: $\exists\, b$ s.t. $\forall x$, $|f(x)| < b$

Know: $\forall \varepsilon, \exists \delta$ s.t. if $x \in V_\delta(c)$ then $|g(x)| < \varepsilon$

wts: $\forall \varepsilon, \exists \delta'$ s.t. if $x \in V_{\delta'}(c)$ then $|f(x)g(x)| < \varepsilon$

Since $f(x)$ is at most b; if $|g(x)| < \frac{\varepsilon}{b}$ then $|f(x)g(x)| < \varepsilon$

if I set $\delta(\varepsilon) = \delta(\frac{\varepsilon}{b})$ then I do have $|g(x)| < \frac{\varepsilon}{b}$ $\forall x \in V_{\delta'}(c)$

Example: $id: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is contin. $id(0) = 0$

$ind_{\mathbb{Q}}$ is bounded so $ind_{\mathbb{Q}} \cdot id: x \mapsto \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ x & \text{if } x \in \mathbb{Q} \end{cases}$

is contin. at $c = 0$; it is discontin. everywhere else

Example: $x \longmapsto \begin{cases} \sin\left(\frac{1}{x}\right)\cdot x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous at 0. (is continuous every where else)

Example: Thomae function

$$x \longmapsto \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases} \qquad n \in \mathbb{Z}_{>0},\ m \in \mathbb{Z} \quad \gcd(m,n) = 1$$

is continuous on $\mathbb{R} - \{0\}$
discontinuous on $\mathbb{Q}$

Theorem: if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous then $f \circ g$ is continuous.
$(f \circ g)(x) = f(g(x))$

→ More generally, if g is continuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c.

proof: Given $\varepsilon > 0$, find $\delta > 0$ s.t. $f(V_\delta(g(c))) \subset V_\varepsilon(f(g(c)))$
then set $\varepsilon' := \delta$, find δ' s.t. $g(V_{\delta'}(c)) \subset V_\varepsilon(g(c))$

→ So $(f \circ g)(V_{\delta'}(c)) \subset f(V_{\varepsilon'}(g(c))) \subset V_\varepsilon((f \circ g)(c))$

example: $\sin(\cos(x))$ is contin.

Theorem: if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at c then $(f+g)$ is contin. at c

proof: Given ε, find $\delta = \min\left(\delta_f\left(\frac{\varepsilon}{2}\right), \delta_g\left(\frac{\varepsilon}{2}\right)\right)$ s.t. $f(V_\delta(c)) \subset V_{\varepsilon/2}(f(c))$ and $g(V_\delta(c)) \subset V_{\varepsilon/2}(g(c))$
→ Then $(f+g)(V_\delta(c)) \subset V_\varepsilon((f+g)(c))$

Example: $x + \sin(x)$

Theorem: if $f, g : \mathbb{R} \to \mathbb{R}$ are contin. at c then $f \cdot g$ is.

proof: if g is contin. at c then $\overset{= g'}{(g - g(c))} : x \longmapsto g(x) - g(c)$ is continuous and $g'(c) = 0$

→ if f is continuous at c then f is bounded in a nbhd of c

Lemma: if $a \in \mathbb{R}$ $x \longmapsto ax$ is continuous

Version of earlier prop: then $f \cdot g'$ is continuous at c ✱

$(f \cdot g)(x) = \underbrace{(f \cdot g')(x)}_{\text{contin. by ✱}} + f(x) \cdot g(c) \rightarrow (h \circ f)(x)$ where $h(x) = ax$ w/ $a = g(c)$ So the whole thing is a composite of contin. functions.

proof: set $\delta = \frac{\varepsilon}{|a|}$ if $a \neq 0$ if $a = 0$ then $ax = 0$ is a constant function.

Example: $\forall N \in \mathbb{N}$ $x \mapsto x^N$ is continuous. proof: $x^N = x^{N-1} \cdot x$ by induction in N, x^{N-1} is contin. x is continuous so x^N is a product of cont. functions

Example: $x \mapsto \frac{1}{x}$ $\mathbb{R}-\{0\} \to \mathbb{R}$ is continuous on its domain.

$\longrightarrow$ $x \cdot \sin\left(\frac{1}{x}\right)$ is continuous at $x \neq 0$

Theorem: Suppose $A \subset \mathbb{R}$ and $f: A \to \mathbb{R}$ is continuous and $c \in cl(A)$ then if $\lim_{x \to c} f(x)$ exists then

$$x \mapsto \begin{cases} f(x) & \text{if } x \in A \\ \lim_{x \to c} f(x) & \text{if } x = c \end{cases}$$

$A \cup \{c\} \to \mathbb{R}$ is continuous. and is the unique constant