# Lecture 20: Continuous Functions

11·Mar·2026

## Continuous Extension:

Suppose $A \subset \mathbb{R}$. $cl(A) = A \cup$ all cluster pts of $A$ e.g. $A = \mathbb{R} - \{0\}$ $cl(A) = \mathbb{R}$ $cl((a,b)) = [a,b]$

Suppose given $f: A \to \mathbb{R}$ continuous on $A$; then ther is at most one function $g: cl(A) \to \mathbb{R}$ s.t.

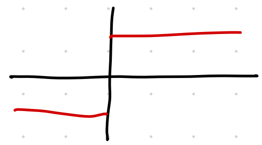
* $g$ is continuous on $cl(A)$
* $g(x) = f(x)$ if $x \in A$

indeed $g$ exists $\iff \forall c \in cl(A)$ $\lim_{x\to c} f(x)$ exists.

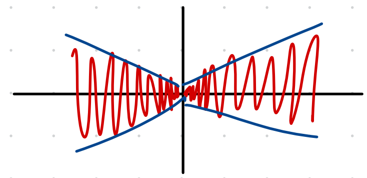
E.g.: $A = \mathbb{R} - \{0\}$


$f(x) = \mathrm{sign}(x) = \begin{cases} +1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$

there does not exist a continuous extension to $cl(A)$



E.g.: $A = \mathbb{R} - \{0\}$ $f(x) = x\,\mathrm{sign}\left(\frac{1}{x}\right)$ $g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous on $\mathbb{R}$



if $g$ exists, then $g(x) = f(x)$ if $x \in A$. If $x \in cl(A)$ then $\because g$ is assumed continuous, we must have:

$$g(c) = \lim_{x\to c} g(x) = \lim_{x\to c} f(x)$$

**Theorem:**

Suppose $f: \mathbb{R} \to \mathbb{R}$ s.t. 1). $f$ is monotonic
2). $f$ is bijective (preserves the order)
then $f$ is continuous

E.g.: $x \mapsto \frac{1}{x}$ $(0,\infty) \to (0,\infty)$

$f: (a,b) \to (a',b')$ $\begin{matrix} a<b \\ a'<b' \end{matrix}$ allow $\begin{matrix} a=-\infty \\ b=+\infty \end{matrix}$

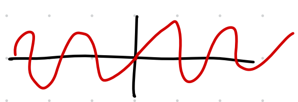
proof: WTS: $\forall\ c \in (a,b)$, $\forall\ \varepsilon > 0$ $\exists\ \delta > 0$ s.t. $V_\varepsilon(f(c)) \supset f(V_\delta(c))$.

for any function $f: A \to M$ ($A, M$ are sets) if $B \subset A$ then $f(B) := \{f(x) \mid x \in B\} \subset M$ "image of B under f"

if $N \subset M$ then $f^{-1}(N) := \{x \in A \mid f(x) \in N\}$ "inverse image (preimage) of N under f"

! In general, $f(f^{-1}(N)) \neq N$ and $f^{-1}(f(B)) \neq B$

E.g.: $A = B = \mathbb{R}$ $f(x) = \sin(x)$



close enough

$f((-\infty,\infty)) = [-1,1]$ $f^{-1}([-1,1]) = \mathbb{R}$

$f^{-1}([-2,2]) = \mathbb{R}$ $f(\mathbb{R}) = [-1,1]$

Do have: $f(f^{-1}(N)) \subset N$ $f^{-1}(f(B)) \supset B$

moreover we do have $f(B) \subset N \iff B \subset f^{-1}(N)$

→ If $f$ happens to be bijective then we have $f^{-1}: M \to A$; could worry if "$f^{-1}(N)$" means two things in fact in this case both interpretations give the same subset of $A$. Moreover, it is true in this case that $ff^{-1}(N) = N$ $f^{-1}f(B) = B$.

→ $V_\varepsilon(f(c)) \supset f(V_\delta(c))$ equivalent to showing $\forall c, \forall \varepsilon, \exists \delta$ s.t. $f^{-1}(V_\varepsilon(f(c))) \supset V_\delta(c)$

$V_\varepsilon(f(c)) = (\underbrace{f(c)-\varepsilon}_{x}, \underbrace{f(c)+\varepsilon}_{y})$ if $\varepsilon$ is small enough $a < x$ $y < b$; if $a < x < y < b$ look at $(x,y) \subset (a,b)$ then set

$x < y$ $\varepsilon := \frac{y-x}{2}$ $c := f^{-1}\left(\frac{x+y}{2}\right)$, then $(x,y) = V_\varepsilon(f(c))$ → if $f$ is monotone increasing then

WTS: $\forall x, y$ with $a < x < y < b$, $\exists \delta$ s.t. $f^{-1}((x,y)) \supset V_\delta(c)$ $c := \frac{x+y}{2}$

$f^{-1}(x) < \underbrace{f^{-1}\left(\frac{x+y}{2}\right)}_{c} < f^{-1}(y)$

→ So pick $\delta = \min(c - f^{-1}(x), f^{-1}(y) - c)$ then win ∎

**Example:** * $\frac{1}{x}$ $(0,\infty) \longrightarrow (0,\infty)$ is continuous * $\exp: (-\infty,\infty) \longrightarrow (0,\infty)$ is continuous

* $\log: (0,\infty) \longrightarrow (-\infty,\infty)$ is continuous

**Definition:** Let $f: A \longrightarrow \mathbb{R}$, $f$ is bounded if $f(A) \subset \mathbb{R}$ is a bounded subset

**non-e.g.:** $x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is unbounded

**Theorem:** Suppose $f: [a,b] \longrightarrow \mathbb{R}$ is continuous. Then $f$ is bounded.

**proof:** Suppose $f$ is unbounded. Then I can find a sequence $x_n$ in $[a,b]$ s.t. "$|f(x)| \to \infty$"
I mean: $\forall n$, can find $x_n \in [a,b]$ s.t. $|f(x_n)| > n$

Using the Bolzano–Weierstrauss I can find a subsequence $y_n \subset x_n$ s.t. $\lim_{x\to\infty} y_n$ exists (b/c although $f(x_n)$ is unbounded, $x_n$ is bounded)
Since $x_n$ stays in $[a,b]$, $y_n$ stays in $[a,b]$ so $\lim_{n\to\infty} y_n$ is in $[a,b]$
If $f$ is continuous then $|f(\lim_{n\to\infty} y_n)| = \lim_{n\to\infty} |f(y_n)|$ so $|f(y_n)|$ converges to some finite value contradicts that $y_n$ is a subsequence of $x_n$, and $|f(x_n)|$ grows uncontrollably.

**Corollary:** $\sup f([a,b])$ exists inf " " "