

Lecture 21: IVT

13. Mar. 2026

Last time: Suppose $I=[a,b]$ is a closed interval and $f:I \rightarrow \mathbb{R}$ is a contin function then f is bounded

Corollary: $f(I)$ has a supremum.

Theorem: In fact, $f(I)$ has a maximum. Equiv: $\sup(f(I)) \in f(I)$

proof: Can find a sequence y_n in $f(I)$ s.t. $\lim_{n \rightarrow \infty} y_n = \sup(f(I))$

- Can find $x_n \in I$ s.t. $y_n = f(x_n)$
- This seq of x_n 's is bounded and hence and has a convergent subseq w_n
- $\lim_{n \rightarrow \infty} w_n \in I$ f is continuous so $f(\lim_{n \rightarrow \infty} w_n) = \lim_{n \rightarrow \infty} f(w_n)$ but $f(w_n)$ is a subseq of $f(x_n) = y_n$
- $\lim_{n \rightarrow \infty} y_n = \sup(f(I))$ since $f(w_n) \subseteq f(x_n)$ have the same limit. ■

Theorem: (Intermediate Value Theorem)

Let $f:[a,b] \rightarrow \mathbb{R}$ be continuous then $f(I) = [\inf(f(I), \sup(f(I))]$

equivalently, I'm claiming that if $if \leq y \leq \sup$ then $\exists x \in [a,b]$ s.t. $f(x) = y$

Simplified IVT: suppose $f:[a,b] \rightarrow \mathbb{R}$ continuous and $f(a) < 0$ and $f(b) > 0$.

→ Then $\exists x \in [a,b]$ s.t. $f(x) = 0$

Proof of Complicated IVT using simplified IVT

- ① if $\inf(f(I)) = \sup(f(I))$ then trivial
- ② if not equal, pick a', b' s.t. $f(a') = \inf(f(I))$ $f(b') = \sup(f(I))$
Up to replacing $f(x)$ by $f(-x)$, can ensure $a' < b'$
- ③ up to replacing $f(x)$ by $f(x) - y$, can assume $y = 0$ → restrict f to $[a', b'] \subseteq [a, b]$

Now I'll apply the simplified version.

proof of simplified IVT: Inspect $f(\frac{a+b}{2})$

→ If it happens to be 0, done.

Else either > 0 or < 0

→ In this case, exactly one of the subintervals has $f(\text{left end pt}) < 0$ and $f(\text{right end pt}) > 0$ and pick that subinterval and repeat.

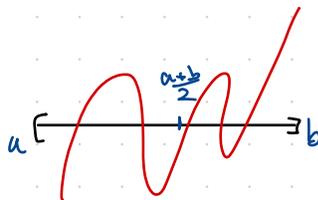
→ If this algorithm runs forever, call a_n the sequence of left end points and b_n sequence of right endpoints.

→ These sequences both converge (monotone bounded).

→ $|b_n - a_n| = \frac{a+b}{2^n} \rightarrow 0$ so $\lim a_n = \lim b_n$; final step is to say the following.

$$f(a_n) < 0 \quad \forall n \implies \lim f(a_n) \leq 0 \quad \rightarrow \text{so } f(\lim a_n) = 0$$

$$f(b_n) > 0 \quad \forall n \implies \lim f(b_n) \geq 0$$



Recall: if $f: \mathbb{R} \rightarrow \mathbb{R}$ is monotonic and bijective then f is continuous.

Theorem: Suppose $f: [a, b] \rightarrow \mathbb{R}$ is ^{strictly} monotonic and continuous \rightarrow means if $x_1 < x_2$ then $f(x_1) < f(x_2)$ and continuous. Then $f: [a, b] \rightarrow f([a, b]) = [f(a), f(b)]$ is bijective.

proof: strict monotone \implies injective IVT \implies surj

\rightarrow Suppose $f: [a, b] \rightarrow \mathbb{R}$ is monotone increasing

\rightarrow Suppose $a < c < b$. Look at $\lim_{x \rightarrow c} f(x) = \sup\{f(x) \mid a \leq x < c\}$ $\lim_{x \rightarrow c} f(x) = \inf\{f(x) \mid x > c\}$

$\lim_{x \rightarrow c} f(x) \leq f(c) \leq \lim_{x \rightarrow c} f(x) \rightarrow$ so f is continuous $\iff \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} f(x)$

\rightarrow Define $j_f(c) := \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} f(x)$ $\{c \text{ s.t. } f \text{ is discontinuous}\} = \{c \text{ s.t. } j(c) \neq 0\}$

Theorem: (f monotone) This set is countable.

proof: Define $D_k := \{c \mid j(c) > 1/k\}$ then $D = \bigcup_{k=1}^{\infty} D_k$

\rightarrow If $c_1, \dots, c_n \in [a, b]$ then $\sum_{i=1}^n j(c_i) \leq f(b) - f(a)$ so $|D_k| \leq k(f(b) - f(a)) < \infty$
hence D is countable union of finite sets so countable.