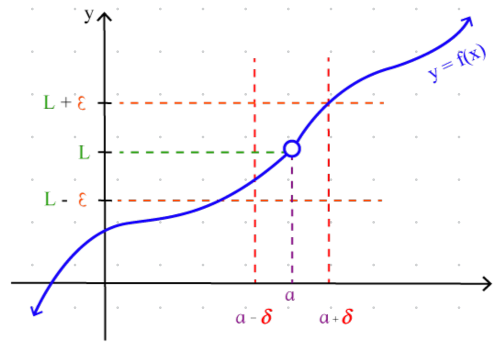
Lecture 22: Uniform & Lipschitz Continuity

16. March. 2026

- $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ → continuous at $a \in A$ if: $\forall \varepsilon > 0, \exists \delta$ s.t. $|x-a| < \delta \implies |f(x) - f(a)| < \varepsilon$
- f is cont. on A if f is cont. at all $a \in A$

Let $f(x)$ be a function defined on the interval that contains $x = a$.
Then $\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there exists some real number $\delta > 0$ so that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$



Uniform and Lipschitz Continuity (5.4)

Definition: $f: A \to \mathbb{R}$ is uniformly continuous on A if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, a \in A$ $|x-a| < \delta \implies |f(x) - f(a)| < \varepsilon$

Definition: if $f: A \to \mathbb{R}$ is Lipschitz continuous if $\exists K > 0$ s.t. $\forall x, a \in A$ $|f(x) - f(a)| \leq K|x-a|$

Theorem 5.4.5: Lipschitz continuity $\implies$ uniform continuity

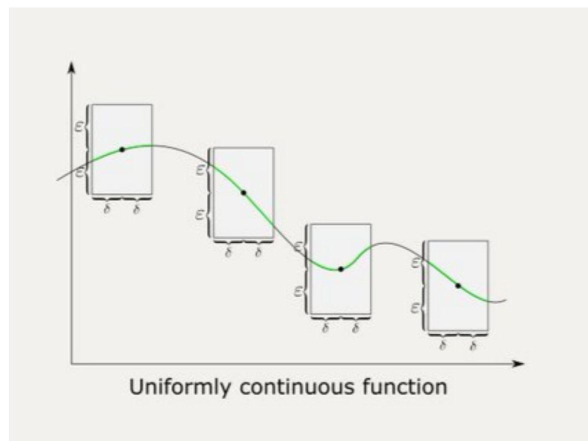
Proof: Suppose Lipschitz continuous, then

$|f(x) - f(a)| \leq K|x-a|$

$|f(x) - f(a)| < \varepsilon$ → Choose $\delta = \frac{\varepsilon}{K}$, then

$|f(x) - f(a)| \leq K|x-a| < K \frac{\varepsilon}{K} = \varepsilon$

hence $|x-a| < \frac{\varepsilon}{K}$ ■

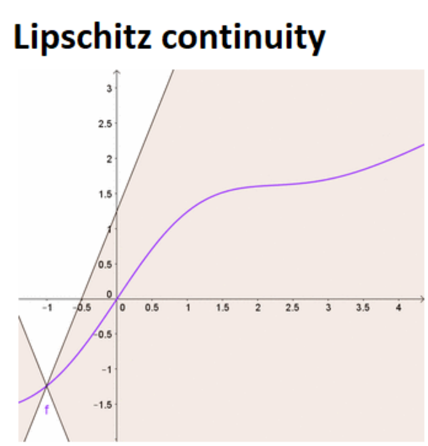


Observation: Uniform cty $\implies$ cty

Lemma 5.4.2: $f: A \to \mathbb{R}$. $A \subseteq \mathbb{R}$ TFAE:

1) f is not uniformly cts on A
2) $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0, \exists x_\delta, a_\delta \in A$ s.t. $|x_\delta - a_\delta| < \delta$ & $|f(x_\delta) - f(a_\delta)| \geq \varepsilon_0$
3) $\exists \varepsilon_0 > 0$ and two sequences (x_n) and (a_n) in A with $\lim(x_n - a_n) = 0$ & $|f(x_n) - f(a_n)| \geq \varepsilon_0$, $\forall n \geq 0$

Theorem 5.4.3: let I be a closed and bounded interval ($I = [a,b]$) $f: I \to \mathbb{R}$ is cont. on $I \iff$ it is uniformly cont. on I.



proof: (by contradiction) Assume f is continuous but not uniformly cont. on I

→ by the Lemma $\implies \exists$ sequences (x_n) and (a_n) s.t. $\lim(x_n - a_n) = 0$

$\forall \varepsilon > 0, \exists N$ s.t. $|x_n - a_n| < \varepsilon, \forall n \geq N$

I can choose $\varepsilon = 1/n$

and $|f(x_n) - f(a_n)| \geq \varepsilon_0$ $\forall n$

$\because I$ is bounded $\implies (x_n)$ & (a_n) are bounded sequences

→ B.W. Theorem $\implies \exists (x_{n_k})$ & (a_{n_k}) subsequences that converge

$(x_{n_k}) \to z$ $(a_{n_k}) \to b$

→ Now I is closed $\implies a, b \in I$; $\because$ both subsequences converge to something and $\lim(x_n - a_n) = 0 \implies \lim(x_{n_k} - a_{n_k}) = 0$ also. $\implies z = b$, i.e. the two subsequences converge to the same point in I.

→ $\lim f(x_{n_k}) = f(\lim(x_{n_k})) = f(z)$ and similarly $\lim f(a_{n_k}) = f(z) \implies \lim(f(x_{n_k}) - f(a_{n_k})) = 0$

then $\forall \varepsilon > 0$ $|f(x_{n_k}) - f(a_{n_k})| < \varepsilon$ for sufficiently large K → which contradicts f not cts. $\implies f$ uniformly cont. ■

Example: $f(x)=x^2$ on $A=[0,b]$, $b>0$ $\longrightarrow$ $\forall x,a\in[0,b]$; $|f(x)-f(a)|=|x^2-a^2|=|x-a||x+a|\leq 2b|x-a|$

$\Longrightarrow x^2$ is Lipschitz cont. on $[0,b]$

$\Longrightarrow$ uniformly cts & cts

Example: $g(x)=\frac{1}{x}$ on $A=(0,\infty)$ $\longrightarrow$ can make sequence $(x_n)=\left(\frac{1}{n}\right)$ and $(a_n)=\left(\frac{1}{n+1}\right)$, then we have

$g(x_n)-g(a_n)=\frac{1}{1/n}-\frac{1}{1/(n+1)}=n-(n+1)=-1$

$\lim(x_n-a_n)=0$ but $g(x_n)-g(a_n)=-1$ $\forall n$ $\Longrightarrow$ by lemma $g(x)$ not uniformly cts. on $(0,\infty)$

Example: $h(x)=\sqrt{x}$ a). $A=[0,2]$ b). $A=[0,\infty)$

a). $h(x)$ is cts. on $[0,2]$, so by Thm, $h(x)$ is uniformly cts on $[0,2]$, is it Lipschitz?

$\forall x,a\in A$ check $|h(x)-h(a)|=|\sqrt{x}-\sqrt{a}|=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}$ Lipschitz $\Longrightarrow \frac{1}{\sqrt{x}+\sqrt{a}}\leq K$ but could choose $x,a=0$, $K\to\infty$ so not bounded.

b). $A=\underbrace{[0,2]}_{I}\cup\underbrace{[2,\infty)}_{J}$ h is Lipschitz cont. on J

$\Longrightarrow$ h is also uniformly cont. on J so uniformly continuous on the union of intervals.