Lecture 23: More Stuff about Uniform Continuity and some stuff on Monotone functions

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→ by continuous we mean $f: A \subseteq \mathbb{R} \to \mathbb{R}$

Recap: Continuous $\not\Rightarrow$ uniformly continuous $\not\Rightarrow$ Lipschitz continuous

- Continuous → $\forall x \in A, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$
  i.e. for every ε-nbhd you can find a delta-nbhd in the domain
- uniformly continuous → $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$
- Lipschitz continuous → $|f(y) - f(x)| \le K|y-x|$ $K > 0$
  for example if $K=1$ $|f(y)-f(x)| \le |y-x|$

Continuous
$\frac{1}{x}$ on $(0,\infty)$ → not uniformly continuous

uniformly continuous:
$\sqrt{x}$ on $(0,\infty)$ → not Lipschitz contin.

Lipschitz:
$\sin(x)$ on $(0,\infty)$ → Lipschitz cont.
$|\sin(y) - \sin(x)| \le |y-x|$

Example: $\sqrt{x}$ is not U.C.

proof: $\forall \varepsilon > 0$, need δ s.t. $|y-x| < \delta \Rightarrow |\sqrt{y} - \sqrt{x}| < \varepsilon$
take $\delta = \varepsilon^2$ hence $|y-x| < \varepsilon^2$; $|\sqrt{y} - \sqrt{x}|^2 \le |\sqrt{y}^2 - \sqrt{x}^2|$ → check $|\sqrt{a} - \sqrt{b}| \le |a^2 - b^2|$
$|\sqrt{y} - \sqrt{x}|^2 \le |\sqrt{y}^2 - \sqrt{x}^2| = |y-x| < \delta = \varepsilon^2 \Rightarrow |\sqrt{y} - \sqrt{x}| < \varepsilon$

Theorem: $A = [a,b]$ then f is continuous $\Leftrightarrow$ f is uniformly continuous.

Continuous $\leftrightarrow$ $\forall x_n \to x \Rightarrow f(x_n) \to f(x)$
$\cup\!\!\!/$
uniformly continuous $\leftrightarrow$ $(x_n - y_n) \to 0 \Rightarrow (f(x_n) - f(y_n)) \to 0$
$\cup\!\!\!/$
Lipschitz continuous $\leftrightarrow$ $\dfrac{|f(x_n) - f(y_n)|}{|x_n - y_n|}$ is bounded $\circledast$

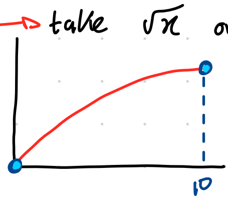
$\circledast$ **Task:** Show that $\sqrt{x}$ is not Lipschitz. Find x_n : y_n s.t. $\dfrac{\sqrt{x_n} - \sqrt{y_n}}{x_n - y_n}$ unbounded,

$$\frac{\sqrt{x_n} - \sqrt{y_n}}{(\sqrt{x_n} + \sqrt{y_n})(\sqrt{x_n} - \sqrt{y_n})} = \frac{1}{\sqrt{x_n} + \sqrt{y_n}}$$

Alternatively can take $x=0$, if Lipschitz then $\exists K$ s.t. $|\sqrt{y} - \sqrt{0}| = |\sqrt{y}| < K|y|$ $\forall y > 0 \Rightarrow \frac{1}{\sqrt{y}} < K$ $\forall y > 0$ → this dosnt work b/c for small y, $\frac{1}{\sqrt{y}}$ explodes, so K is not a bound.

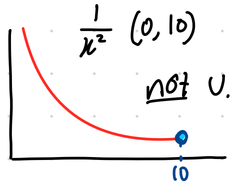
Theorem: $f: (a,b) \to \mathbb{R}$ f is uniform continuous $\Leftrightarrow$ f can be extended to a continuous function $\tilde{f}: [a,b] \to \mathbb{R}$

→ take $\sqrt{x}$ on $(0,10)$



$\frac{1}{x^2}$ $(0,10)$

not U.C.



Corollary: if $f: (a,b) \to \mathbb{R}$ $\lim_{x \searrow a} f$ DNE then ⟹ not U.C. ?

Lemma: if $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is uniformly continuous ⟹ $f(x_n)$ is Cauchy (2) and x_n is cauchy seq. (1)
→ uniformly cont. functions preserve sequences.

proof: (1) $\forall \alpha > 0, \exists N$ s.t. $n,m > N \Rightarrow |x_n - x_m| < \alpha$ → eventually the points get very close to each other.
(2) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$

wts: $\forall \beta > 0, \exists M$ s.t. $n,m > M \Rightarrow |f(x_n) - f(x_m)| < \beta$; step (B) set $\alpha = \delta$, apply (1), then $\exists M$ s.t. $n,m > M$ $|x_n - x_m| < \delta$

step (A) set $\varepsilon = \beta$ in (2), obtain δ s.t. $|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$ → $|f(x_n) - f(x_m)| < \varepsilon$

proof: ($\Leftarrow$) uniform cont. $\Longrightarrow$ cont. on closed interval

($\Longrightarrow$) $\hat{F}(x) = \begin{cases} f(x), & a<x<b \\ \lim\limits_{x \searrow a} f, & x=a \\ \lim\limits_{x \nearrow b} f, & x=b \end{cases}$

$(a,b) \supseteq (x_n)$ Cauchy $x_n \to a$ and hence $f(x_n)$ is Cauchy

$f(x_n) \to L$ in $\mathbb{R}$ $:= \hat{f}(a)$; need $\hat{F}$ is continuous; suppose we have $x_n \to [a,b]$ convergent sequence

$y_n = a$-valued $\to a$

$z_n = (a,b]$-valued $\to a$ $\quad f(y_n) = f(a) = L \quad f(z_n) = f(a) = L$