Lecture 24: Sequence of functions

23. Mar. 2026

Definition:

→ fix $A \subset \mathbb{R}$, a <u>sequence of functions</u> on A is: for each $n \in \mathbb{N}$ a function $f_n : A \to \mathbb{R}$

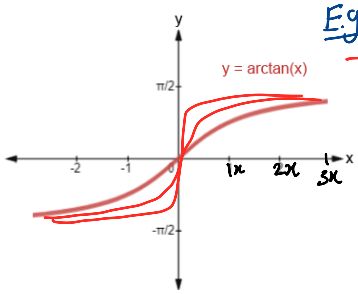
for each $x \in A$, get a sequence of numbers $f_n(a)$.

→ can ask, given a choice of $a \in A$, does $f_n(a)$ converge?

Definition: A sequence of functions f_n <u>converges pointwise</u> if $\forall a \in A$, the sequence of numbers $f_n(a)$ converges. In this case $\lim_{n\to\infty} f_n(a)$ is a specific number depending on a, i.e. we get a function $\lim_{n\to\infty}(f_n)$ pointwise, defined by:

$$\forall a, \quad \lim_{n\to\infty}(f_n(a)) = \Big(\lim_{n\to\infty}^{\text{pointwise}}(f_n)\Big)(a)$$

→ Said another way, the sequence of f_n <u>converges pointwise</u> to f if $\forall a \in A$, $\lim_{n\to\infty} f_n(a) = f(a)$



<u>E.g.:</u> definition: $\arctan(x)$ is the unique solution to $\tan(y) = x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

→ look at sequence $\arctan(nx)$, if $x = 0$ we have

$$\arctan(0x) = \arctan(0) = 0$$

→ if $a > 0$, then na is increasing monotonically and $\arctan(-)$ is monotonic so, $\arctan(na)$ is increasing, but bounded by $\pi/2$ so converges.

→ In fact, if $a > 0$ then

$$\lim_{n\to\infty} \arctan(na) = \pi/2$$

→ If $a < 0$ then $\lim_{n\to\infty} \arctan(na) = -\pi/2$

→ <u>Summary</u>: the sequence $\arctan(nx)$ converges pointwise to $\frac{\pi}{2}\operatorname{sign}(x) = \begin{cases} \pi/2, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\pi/2 & \text{if } x < 0 \end{cases}$

→ <u>Shows</u>: pointwise limit of conts. functions might not be continuous.

<u>Unpacking the definition:</u> $f_n \to f$ pointwise if $\forall a \in A$, $\forall \varepsilon > 0$, $\exists N(\varepsilon, a)$ s.t. if $n > N(\varepsilon, a)$ then $|f_n(a) - f(a)| < \varepsilon$

Definition: The sequence f_n <u>converges uniformly</u> to f if $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t. if $n > N(\varepsilon)$ then $\forall a \ |f_n(a) - f(a)| < \varepsilon$

<u>Obv:</u> uniform convergence $\Longrightarrow$ pointwise convergence

→ failure of uniform convergence means that $\exists \varepsilon > 0$ s.t. $\forall N$ we can find an $x \in A$ s.t. $\exists n > N$ with $|f_n(x) - f(x)| \geq \varepsilon$

→ i.e. put x_N's into a sequence; put f_{n_N} into a subsequence of f

→ <u>Conclude</u>: f_n does not converge uniformly if $\exists \varepsilon$ and $\exists$ sequence x_k and f_{n_k} a subsequence of f_n s.t. $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon$

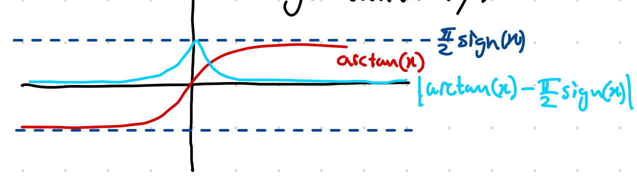
Definition: <u>Lemma:</u> $\|g(x)\| = \|g(nx)\| \quad \forall n \neq 0$

If $g : A \to \mathbb{R}$ is a bounded function. Define the <u>uniform norm</u> aka <u>sup norm</u> aka <u>L^∞ norm</u> of g is $\|g\| := \sup_{a\in A} |g(a)| \in \mathbb{R}$. → $f_n \to f \iff \|f_n - f\| \to 0$

<u>Example:</u> $\arctan(nx) \to \frac{\pi}{2}\operatorname{sign}(x)$ pointwise and now we ask does it converge uniformly?

$$\|\arctan(nx) - \tfrac{\pi}{2}\underbrace{\operatorname{sign}(x)}_{\operatorname{sign}(nx)}\| = \|\arctan(x) - \tfrac{\pi}{2}\operatorname{sign}(x)\| = \tfrac{\pi}{2}$$

<u>no:</u> $\|\arctan(nx) - \frac{\pi}{2}\operatorname{sign}(x)\| \not\to 0$



→ $\|f\|_{L^2} := \int_{x\in\mathbb{R}} |f(x)|^2 dx$ like $\|\vec{v}\| = \left(\sum_i |v_i|^2\right)^{1/2}$

$\|f\|_{L^p} := \left(\int_{x\in\mathbb{R}} |f(x)|^p dx\right)^{1/p}$ like $\left(\sum_i |v_i|^p\right)^{1/p}$